ON THE NON-TRIVIALITY OF THE p-ADIC ABEL-JACOBI IMAGE OF GENERALISED HEEGNER CYCLES MODULO p, II: SHIMURA CURVES

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ABSTRACT. Generalised Heegner cycles are associated to a pair of an elliptic newform and a Hecke character over an imaginary quadratic extension $K/\mathbf{Q}$. The cycles live in a middle dimensional Chow group of a Kuga-Sato variety arising from an indefinite Shimura curve over the rationals and a self product of a CM abelian surface. Let p be an odd prime split in $K/\mathbf{Q}$. We prove the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p over the $\mathbf{Z}_p$-anticylotomic extension of K. The result implies the non-triviality of the generalised Heegner cycles in the top graded piece of the coniveau filtration on the Chow group and proves a higher weight analogue of Mazur's conjecture. In the case of two, the result provides a refinement of the results of Cornut-Vatsal and Aflalo-Nekovář on the non-triviality of Heegner points over the $\mathbf{Z}_p$-anticylotomic extension of K.

CONTENTS

1. Introduction

When a pure motive over a number field is self-dual with root number -1, the Bloch-Beilinson conjecture implies the existence of a non-torsion homologically trivial cycle in the Chow realisation. For a prime p, the Bloch-Kato conjecture implies the non-triviality of the p-adic étale Abel-Jacobi image of the cycle. A natural question is to further investigate the non-triviality of the p-adic Abel-Jacobi image of the cycle.

An instructive set up arises from a self-dual Rankin-Selberg convolution of an elliptic Hecke eigenform and a theta series over an imaginary quadratic extension K with root number -1. In this situation, a natural candidate for a non-torsion homologically trivial cycle is the generalised Heegner cycle. It lives in a middle dimensional Chow group of a fiber product of a Kuga-Sato variety arising from an indefinite Shimura curve and a self product of a CM abelian surface. In the case of weight two, the cycles coincide with the Heegner points. Twists of the theta series by p-power order anticyclotomic characters of K give rise to an Iwasawa theoretic family of generalised Heegner cycles. Under mild hypotheses, we prove the generic non-triviality of the p-adic Abel-Jacobi image of these cycles modulo p. In particular, this implies the generic non-triviality of the cycles in the top graded piece of the coniveau filtration along the $\mathbf{Z}_p$-anticyclotomic extension of K.

In the introduction, for simplicity we mostly restrict to the case of Heegner points.

Let p be an odd prime. We fix two embeddings $\iota_\infty \colon \overline{\mathbf{Q}} \to \mathbf{C}$ and $\iota_p \colon \overline{\mathbf{Q}} \to \mathbf{C}_p$. Let v_p be the p-adic valuation induced by ι_p so that $v_p(p) = 1$.

Let $K/\mathbf{Q}$ be an imaginary quadratic extension as above and $\mathcal{O}$ the ring of integers. As K is a subfield of the complex numbers, we regard it as a subfield of the algebraic closure $\overline{\mathbf{Q}}$ via the embedding ι_∞. Let c be the complex conjugation on $\mathbf{C}$ which induces the unique non-trivial element of $\mathrm{Gal}(K/\mathbf{Q})$ via ι_∞. We assume the following:

(ord) p splits in K.

Let $\mathfrak{p}$ be a prime above p in K induced by the p-adic embedding ι_p. For a positive integer m, let H_m be the ring class field of K with conductor m and $\mathcal{O}_m = \mathbf{Z} + m\mathcal{O}$ the corresponding order. Let H be the Hilbert class field.

Let N be a positive integer such that $p \nmid N$. Let f be an elliptic newform of weight 2, level $\Gamma_0(N)$ and neben-character ϵ. Let $N_\epsilon | N$ be the conductor of ϵ. Let E_f be the Hecke field of f and $\mathcal{O}_{E_f}$ the ring of integers. Let $\mathfrak{P}$ be a prime above p in E_f induced by the p-adic embedding ι_p. Let $\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathcal{O}_{E_f,\mathfrak{P}})$ be the corresponding p-adic Galois representation.

We fix a decomposition $N = N^+ N^-$ where N^+ (resp. N^-) is only divisible by split (resp. ramified or inert) primes in $K/\mathbf{Q}$. We assume the following hypotheses:

(H1) The level N is square-free and prime to the discriminant of K.

(H2) The number of primes dividing N^- is positive and even.
(H3) The conductor N_ϵ divides N^+.

As K satisfies the Heegner hypothesis for N^+, the integer ring $\mathcal{O}$ contains a cyclic ideal $\mathfrak{N}^+$ of norm N^+.
From now, we fix such an ideal $\mathfrak{N}^+$. Let $\mathfrak{N}_\epsilon | \mathfrak{N}^+$ be the unique ideal of norm N_ϵ.

Let $\mathbf{N} : \mathbf{A}_{\mathbf{Q}}^\times / \mathbf{Q}^\times \to \mathbf{C}^\times$ be the norm Hecke character over $\mathbf{Q}$ given by

$$\mathbf{N}(x) = ||x||.$$

Here $|| \cdot ||$ denotes the adelic norm. Let $\mathbf{N}_K := \mathbf{N} \circ N_{\mathbf{Q}}^K$ be the norm Hecke character over K for the relative norm $N_{\mathbf{Q}}^K$. For a Hecke character $\lambda : \mathbf{A}_K^\times / K^\times \to \mathbf{C}^\times$ over K, let $\mathfrak{f}_\lambda$ (resp. ϵ_λ) denote its conductor (resp. the restriction $\lambda|_{\mathbf{A}_{\mathbf{Q}}^\times}$). We say that λ is central critical for f if it is of infinity type (j_1, j_2) with $j_1 + j_2 = 2$ and $\epsilon_\lambda = \epsilon \mathbf{N}^2$.

Let b be a positive integer prime to N. Let $\Sigma_{cc}(b, \mathfrak{N}^+, \epsilon)$ be the set of Hecke characters λ such that:

(C1) λ is central critical for f,
(C2) $\mathfrak{f}_\lambda = b \cdot \mathfrak{N}_\epsilon$.

Let χ be a finite order Hecke character such that $\chi \mathbf{N_K} \in \Sigma_{cc}(b, \mathfrak{N}^+, \epsilon)$. Let $E_{f,\chi}$ be the finite extension of E_f obtained by adjoining the values of χ.

Let B be an indefinite quaternion algebra over $\mathbf{Q}$ of conductor N^-. Let Sh_B be the corresponding Shimura curve of level $\Gamma_1(N^+)$ and J_B the Jacobian of Sh_B. Let f_B be a normalised Jacquet-Langlands transfer of f to Sh_B (cf. §5.1). Let B_f be the abelian variety associated to f_B by the Eichler-Shimura correspondence and $T_f \subset E_f$ an order such that B_f has T_f-endomorphisms. Let $\Phi_f : J_B \to B_f$ be the associated surjective morphism. Let ω_{f_B} be the differential form on Sh_B corresponding to f_B. We use the same notation for the corresponding one form on J_B. Let $\omega_{B_f} \in \Omega^1(B_f/E_f)^{T_f}$ be the unique one form such that $\Phi_f^*(\omega_{B_f}) = \omega_{f_B}$. Here $\Omega^1(B_f/E_f)^{T_f}$ denotes the subspace of 1-forms given by

$$\Omega^1(B_f/E_f)^{T_f} = \left\{ \omega \in \Omega^1(B_f/E_f) | [\lambda]^* \omega = \lambda \omega, \forall \lambda \in T_f \right\}.$$

Let A_b be an abelian surface with endomorphisms by $\mathcal{O}_b = \mathbf{Z} + b\mathcal{O}$, defined over the ring class field H_b. Let ι_{N^+} denote a level structure of $\Gamma_1(N^+)$-type on $A_b[\mathfrak{N}^+]$. In view of the moduli interpretation of the Shimura curve (cf. §2.2), we thus obtain a point $(A_b, \iota_{N^+}) \in Sh_B(H_{N^+b})$. Here we suppress additional data needed in the moduli interpretation for simplicity. Let $\Delta_b \in J_B(H_{N^+b})$ be the corresponding Heegner point on the modular Jacobian. Here we cohomologically trivialise the Heegner point as in Zhang's work, for example [51]. We regard χ as a character $\chi : \mathrm{Gal}(H_{N^+b}/K) \to E_{f,\chi}$. Let $G_b = \mathrm{Gal}(H_{N^+b}/K)$. Let H_χ be the abelian extension of K cut out by the character χ. To the pair (f, χ), we associate the Heegner point $P_f(\chi)$ given by

$$(1.1) \qquad P_f(\chi) = \sum_{\sigma \in G_b} \chi^{-1}(\sigma) \Phi_f(\Delta_b^\sigma) \in B_f(H_\chi) \otimes_{T_f} E_{f,\chi}.$$

To consider the non-triviality of the Heegner points $P_f(\chi)$ as χ varies, we can consider the non-triviality of the corresponding p-adic formal group logarithm. The restriction of the p-adic formal group logarithm arising from the one form ω_{B_f} gives a homomorphism $\log_{\omega_{B_f}} : B_f(H_\chi) \to \mathbf{C}_p$. We

extend it to $B_f(H_\chi) \otimes_{T_f} E_{f,\chi}$ by $E_{f,\chi}$-linearity.

We now fix a finite order Hecke character η such that $\eta \mathbf{N_K} \in \Sigma_{cc}(1, \mathfrak{N}^+, \epsilon)$. Let $H_{N^+ p^\infty} = \bigcup_{n \geq 0} H_{N^+ p^n}$ be the ring class field of conductor $N^+ p^\infty$. Let $K_\infty \subset H_{N^+ p^\infty}$ be the anticyclotomic $\mathbf{Z}_p$-extension of K. In the above notation, we have $G_{p^n} = \mathrm{Gal}(H_{N^+ p^n}/K)$. Let Γ be the $\mathbf{Z}_p$-quotient of $\varprojlim G_{p^n}$. Let $\mathfrak{X}_0$ be the subgroup of finite order characters of the group $\mathrm{Gal}(K_\infty/K) \simeq \mathbf{Z}_p$. Let $\mathfrak{X}$ denote the set of anticyclotomic Hecke characters over K factoring through Γ. For $\nu \in \mathfrak{X}_0$, let $G(\nu)$ denote the Gauss sum associated to ν considered as a primitive character. We consider the non-triviality of $G(\nu^{-1}) \log_{\omega_{B_f}}(P_f(\eta\nu))$ modulo p, as $\nu \in \mathfrak{X}_0$ varies.

Our result is the following.

Theorem A. Let the notation be as above. Let $f \in S_2(\Gamma_0(N), \epsilon)$ be an elliptic newform and η a finite order unramified Hecke character over K such that $\eta \mathbf{N_K} \in \Sigma_{cc}(1, \mathfrak{N}^+, \epsilon)$. In addition to the hypotheses (ord), (H1), (H2) and (H3), suppose that

(irr) the residual Galois representation ρ_f modulo p is absolutely irreducible.

Then, we have

$$\liminf_{\nu \in \mathfrak{X}_0} v_p \left(G(\nu^{-1}) \log_{\omega_{B_f}}(P_f(\eta\nu)) \right) = 0.$$

In particular, for $\nu \in \mathfrak{X}_0$ with sufficiently large p-power order the Heegner points $P_f(\eta\nu)$ are non-zero in $B_f(H_{\eta\nu}) \otimes_{T_f} E_{f,\eta\nu}$.

In fact, we show that the same conclusions hold when $\mathfrak{X}_0$ is replaced by any of its infinite subset. For analogous non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p, we refer to §6.2.

The proof of Theorem A is based on the vanishing of the μ-invariant of an anticyclotomic Rankin-Selberg p-adic L-function and a recent p-adic Waldspurger formula due to Brooks.

We now describe the result regarding the μ-invariant. Associated to the pair (f, η), an anticyclotomic Rankin-Selberg p-adic L-function $L_p(f, \eta) \in \overline{\mathbf{Z}}_p[\![\Gamma]\!]$ is constructed in [6]. It is characterised by the interpolation formula

$$(1.2) \qquad\qquad \widehat{\lambda}(L_p(f, \eta)) \doteq L(f, \eta\lambda \mathbf{N_K}, 0).$$

Here λ is an unramified Hecke character over K with infinity type $(m, -m)$ for $m \geq 0$ and $\widehat{\lambda}$ its p-adic avatar. The notation "$\doteq$" denotes that the equality holds up to well determined periods. Here we only mention that the periods crucially depend on the Jacquet-Langlands transfer f_B and the underlying Shimura curve.

Our result regarding the non-triviality of the p-adic L-function $L_p(f, \eta)$ is the following.

Theorem B. Let the notation be as above. Let $f \in S_2(\Gamma_0(N), \epsilon)$ be an elliptic newform and η a finite order unramified Hecke character over K such that $\eta \mathbf{N_K} \in \Sigma_{cc}(1, \mathfrak{N}^+, \epsilon)$. Suppose that the hypotheses (ord), (H1), (H2), (H3) and (irr) hold. Then, we have
$$\mu(L_p(f, \eta)) = 0.$$

We now describe the strategy of the proofs. Some of the notation used here is not followed in the rest of the article. The characters in $\mathfrak{X}_0$ are outside the range of interpolation for the p-adic L-function $L_p(f, \eta)$ and these values basically equal the p-adic formal group logarithm of Heegner points. This is based on the p-adic Waldspurger formula in [6]. A phenomena of this sort was first found by Rubin in the CM case (*cf.* [43]) and recently by Bertolin-Darmon-Prasanna in the general case (*cf.* [2]). As $\mathfrak{X}_0$ is a dense subset of characters of Γ, the p-adic Waldspurger formula reduces Theorem A to Theorem B. We prove the later based on a strategy of Hida. This strategy was introduced in [26]. Hida proves the vanishing of the μ-invariant of a class of anticyclotomic Katz p-adic L-functions in [26]. Let $G_{f,\eta} \in \overline{\mathbf{Z}}_p[\![T]\!]$ be the power series expansion of the measure $L_p(f, \eta)$ regarded as a p-adic measure on $\mathbf{Z}_p$ with support in $1 + p\mathbf{Z}_p$, i.e.

$$(1.3) \qquad G_{f,\eta} = \int_{1+p\mathbf{Z}_p} (1+t)^y dL_p(f,\eta)(y) = \sum_{k \in \mathbf{Z}_{\geq 0}} \Big(\int_{1+p\mathbf{Z}_p} \binom{y}{k} dL_p(f,\eta)(y) \Big) t^k.$$

The starting point is the fact that there are modular forms $(f_i)_{i=1}^m$ on the Shimura curve Sh_B such that

$$(1.4) \qquad G_{f,\eta} = \sum_{i=1}^m a_i \circ (f_i(t)).$$

Here $f_i(t)$ is the t-expansion of f_i around a well chosen CM point x corresponding to the trivial ideal class in $\mathrm{Pic}(\mathcal{O})$ on the Shimura curve Sh_B. Moreover, a_i is an automorphism of the deformation space of x in Sh_B such that the a_i's are mutually irrational. The modular forms f_i's are closely related to the Jacquet-Langlands transfer f_B. Based on Chai's study of Hecke-stable subvarieties of a Shimura variety, we prove the linear independence of $(a_i \circ f_i)_{i=1}^m$ modulo p. The independence is an analogue of Ax-Lindemann-Weierstrass conjecture for the mod p reduction of the Shimura curve Sh_B. The proof relies on Chai-Oort rigidity principle that a Hecke stable subvariety of a mod p Shimura variety is a Shimura subvariety. The principle is a mod p analogue of Andre-Oort conjecture for self products of the Shimura curve. An analogue of Ax-Lindemann-Weierstrass of this sort was originally found by Hida in the Hilbert modular case (*cf.* [26]). We closely follow Hida's approach. The assumption (irr) plays a key role in the independence as it implies the non-constancy of f_i's modulo p. In view of the independence and (1.4), it follows that $\mu(L_p(f, \eta)) = \min_i \mu(f_i(t)) = \min_i \mu(f_i)$. Based on our optimal choice of the Jacquet-Langlands transfer f_B and the p-integrality criterion in [42], we deduce that $\min_i \mu(f_i) = 0$. This finishes the proof. We would like to emphasize our perception that the independence lies at the heart of the proof of Theorem B.

"In particular" part of the Theorem A was conjectured by Mazur in the early 1980's (*cf.* [37]). It was proven by Cornut-Vatsal and Aflalo-Nekovář in the mid and late 2000's, respectively (*cf.* [15], [47], [16], [17] and [1]). We give a new approach and as far as we know the theorem is a first result regarding the non-triviality of the p-adic formal group logarithm of the Heegner points modulo p. It seems suggestive to compare our approach with the earlier approach. In the earlier approach, Ratner's theorem on ergodicity of torus actions is fundamental. As indicated above, our approach fundamentally relies on Chai's theory of Hecke stable subvarieties of a mod p Shimura variety. It is rather surprising that we have these quite different approaches for the same characteristic zero

non-triviality. A speculation along these lines was expressed in [48]. It seems interesting that the ergodic nature of the earlier approach is still present in our approach albeit in a more geometric form. For a more detailed comparison, we refer to [12]. Before the p-adic Waldspurger formula, the non-triviality and Hida's strategy appeared to be complementary. The formula also allows a rather smooth transition to the higher weight case.

As far as we know, higher weight analogue of Theorem A is a first general result regarding the non-triviality of generalised Heegner cycles. In particular, the non-triviality implies that the Griffiths group of the fiber product of a Kuga-Sato variety arising from the Shimura curve and self product of a CM abelian surface has infinite rank over $\overline{\mathbf{Q}}$ (*cf.* remark (2) following Theorem 6.2). An analogue of such a result for the Griffiths group of the Kuga-Sato variety is due to Besser (*cf.* [4]). In [4], the approach is based on the generic non-triviality of classical Heegner cycles over a class of varying imaginary quadratic extensions. Regarding generalised Heegner cycles in the case of modular curves, the only earlier known result seems to be the non-triviality of several examples of such families in [3]. Based on the Cornut-Vatsal approach, Howard has proven a related characteristic zero non-triviality of classical Heegner cycles in the case of modular curves in [29] and [30].

Theorem A and its higher weight analogue has various arithmetic applications. The non-triviality of the p-adic Abel-Jacobi image along the $\mathbf{Z}_p$-anticyclotomic extension is a typical hypothesis while working with an Euler system, for example [38] and [23]. The hypothesis typically plays a key role in bounding the size of relevant Selmer groups. Cornut-Vatsal's non-triviality results have been applied by Nekovář in his proof of the parity conjecture for a class of Selmer groups (*cf.* [39], [40], [41] and references therein). It seems that the higher weight analogue of Theorem A can have potential applications to the parity conjecture for Selmer groups associated to a more general Rankin-Selberg convolution. As mentioned before, the non-triviality gives an evidence for the refined Bloch-Beilinson conjecture. Some other arithmetic consequences of the "In particular" part of Theorem A are well documented in the literature, for example [15], [47], [1] and [23].

Theorem B is perhaps of independent interest as well. It is an input in the ongoing work of Jetchev, Skinner and Wan on the p-adic Birch and Swinnerton-Dyer conjecture (*cf.* [33]). The result in turn is expected to play a key role in the ongoing work of Bhargava, Skinner and Zhang on the Birch and Swinnerton-Dyer conjecture for a large proportion of elliptic curves over the rationals.

Hida's strategy has been influential in the study of non-triviality of various arithmetic invariants modulo p. For example, the non-triviality results in [26], [7], [31], [32], [8] and [11] are based on the Hilbert modular independence in [26]. The strategy and especially its geometric aspects have been further explored and refined in [9].

Recently, the p-adic Waldspurger formula has been generalised to modular forms on Shimura curves over a totally real field (*cf.* [36]). In [10], we consider an analogue of the independence for quaternionic Shimura varieties over a totally real field. In the near future, we hope to consider an analogous non-triviality of generalised Heegner cycles over a CM field. In a forthcoming article [13], we consider an (l, p)-analogue of the results. Such an analogue in the case of modular curves is considered in [11].

The article is perhaps a follow up to [2] and particularly, [26]. We refer to them for a general introduction. In the exposition, we often suppose that the reader is familiar with them, particularly [26].

The article is organised as follows. In §2, we describe generalities regarding the Shimura curve.

In §3, we consider the Serre-Tate deformation space of an ordinary closed point on the Shimura curve. Some parts of §3 perhaps logically come before §2. For example, the conclusion of §3.1 is needed in the beginning of §2.4. We suggest the reader to proceed accordingly. For convenience, we maintain the current ordering. In §4, we prove the independence of mod p modular forms. In §5, we prove the vanishing of the μ-invariant of a class of anticyclotomic Rankin-Selberg p-adic L-functions. In §6, we prove the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p over the $\mathbf{Z}_p$-anticyclotomic extension.

We refer to the survey [12] for an expository account of the article.

Notation We use the following notation unless otherwise stated.

Let D_K be the different (resp. discriminant) of $K/\mathbf{Q}$. Let v be a place of $\mathbf{Q}$ and w be a place of K above v. Let $\mathbf{Q}_v$ be the completion of $\mathbf{Q}$ at v, ϖ_v an uniformiser and $K_v = \mathbf{Q}_v \otimes_{\mathbf{Q}} K$.

For a number field L, let $\mathbf{A}_L$ be the adele ring, $\mathbf{A}_{L,f}$ the finite adeles and $\mathbf{A}_{L,f}^{\square}$ the finite adeles away from a finite set of places $\square$ of L. Let G_L be the absolute Galois group of L and $\mathrm{rec}_L : \mathbf{A}_L^{\times} \to G_L^{ab}$ the geometrically normalized reciprocity law.

2. Shimura curves

In this section, we describe generalities regarding Shimura curves arising from indefinite quaternion algebras over the rationals.

2.1. **Setup.** In this subsection, we recall a basic setup regarding Shimura curves. We occasionally follow [6]

Let the notation and hypotheses be as in the introduction. In particular, B denotes the indefinite quaternion algebra over $\mathbf{Q}$ of conductor N^-. Let $\mathcal{O}_B$ be a maximal order in [6, §2.1].

Let p_0 be an auxiliary prime such that:

(A1) For a prime l, the Hilbert symbol $(p_0, N^-)_l = 1$ if and only if $l|N^-$.
(A2) All primes dividing pN^+ split in the real quadratic field $M_0 := \mathbf{Q}(\sqrt{p_0})$.

The choice of p_0 determines a Hashimoto model for B as follows. The algebra B has a basis $\{1, t, j, tj\}$ as a vector space with $t^2 = -N^-$, $j^2 = p_0$ and $tj = -jt$ (*cf.* [6, §2.1]). Moreover, the **Z**-span of the basis is contained in a unique maximal order in $\mathcal{O}_B$. Let $\dagger$ be an involution on B given by $b \mapsto b^\dagger = t^{-1}\overline{b}t$ for $b \in B$. Here $b \mapsto \overline{b}$ is the main involution of B.

Let $G_{/\mathbf{Q}}$ be the algebraic group $B^\times$ and $h_0 : Res_{\mathbf{C}/\mathbf{R}}\mathbb{G}_m \to G_{/\mathbf{R}}$ be the morphism of real group schemes arising from

$$a + bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where $a + bi \in \mathbf{C}^\times$. Let X be the set of $G(\mathbf{R})$-conjugacy classes of h_0. We have a canonical isomorphism $X \simeq \mathbf{C} - \mathbf{R}$. The pair (G, X) satisfies Deligne's axioms for a Shimura variety. It gives rise to a tower $(Sh_K = Sh_K(G, X))_K$ of smooth proper curves over $\mathbf{Q}$ indexed by open compact subgroups K of $G(\mathbf{A}_{\mathbf{Q}, f})$. The complex points of these curves are given by

$$(2.1) \qquad\qquad Sh_K(\mathbf{C}) = G(\mathbf{Q})\backslash X \times G(\mathbf{A}_{\mathbf{Q}, f})/K.$$

In what follows, we consider the case when K arises from the congruence subgroup $\Gamma_1(N^+)$. Here $\Gamma_1(N^+)$ denotes the norm one elements in an Eichler order of level N^+ in $\mathcal{O}_B$. We use the notation Sh_B to denote the corresponding Shimura curve.

2.2. p-integral model.
In this subsection, we briefly recall p-integral smooth models of the Shimura curves. We occasionally follow [6].

Let the notation and hypotheses be as in §2.1. The Shimura curve $Sh_{B/\mathbf{Q}}$ represents a functor $\mathcal{F}$ classifying abelian surfaces having multiplication by $\mathcal{O}_B$ along with additional structure (*cf.* [6, §2.2]). A p-integral interpretation of $\mathcal{F}$ leads to a p-integral smooth model of $Sh_{B/\mathbf{Q}}$. A closely related functor $\mathcal{F}^{(p)}$ gives rise to the relevant Shimura variety of level prime to N^-p i. e. a tower of Shimura curves of level prime to N^-p. For later purposes, we consider the Shimura variety.

The functor $\mathcal{F}^{(p)}$ is given by

$$\mathcal{F}^{(p)} : SCH_{/\mathbf{Z}_{(p)}} \to SETS$$

$$(2.2) \qquad\qquad S \mapsto \{(A, \iota, \overline{\lambda}, \eta^{(p)})_{/S}\}/ \sim .$$

Here,

(PM1) A is an abelian surface over S.

(PM2) $\iota : \mathcal{O}_B \hookrightarrow \text{End}_S A$ is an algebra embedding.

(PM3) $\overline{\lambda}$ is a $\mathbf{Z}_{(p)}^+$-polarisation class of a homogeneous polarisation λ such that the Rosati involution of $\text{End}_S A$ maps $\iota(l)$ to $\iota(l^\dagger)$ for $l \in \mathcal{O}_B$.

(PM4) Let $\mathcal{T}^{(N^-p)}(A)$ be the prime to N^-p Tate module $\varprojlim_{(M, N^-p)=1} A[M]$. The notation $\eta^{(p)}$ denotes a full level structure of level prime to N^-p given by an $\mathcal{O}_B$-linear isomorphism

$$\eta^{(p)} : \mathcal{O}_B \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}^{(N^-p)} \simeq \mathcal{T}^{(N^-p)}(A)$$

for $\widehat{\mathbf{Z}}^{(N^-p)} = \prod_{l \nmid N^-p} \mathbf{Z}_l$.

The notation $\sim$ denotes up to a prime to N^-p isogeny.

Theorem 2.1 (Morita, Kottwitz)**.** *The functor $\mathcal{F}^{(p)}$ is represented by a smooth proper pro-scheme* $Sh^{(p)}_{/\mathbf{Z}[\frac{1}{N}]}$. *(cf. [6, §2.2]).*

Let $\mathcal{A}$ be the universal abelian surface.

2.3. **Idempotent.** In this subsection, we describe generalities regarding an idempotent in the ring of algebraic correspondences on the universal abelian surface.

Let the notation and hypotheses be as in §2.1. Let $\epsilon \in \mathcal{O}_B \otimes \mathcal{O}_{M_0}[\frac{1}{2p_0}]$ be the non-trivial idempotent given by

$$\epsilon = \frac{1}{2}\left(1 \otimes 1 + \frac{1}{p_0}j \otimes \sqrt{p_0}\right).$$

Based on the hypotheses (H1) and (H2), we have an optimal embedding

$$\iota_K : K \hookrightarrow B$$

(*cf.* [6, §2.4]).

In view of the moduli interpretation of the Shimura variety $Sh^{(p)}$, we can let ϵ naturally act on the test objects as a correspondence. It follows that ϵ can be regarded as an algebraic correspondence on the universal abelian surface $\mathcal{A}$.

Via the p-adic embedding ι_p, we often regard the idempotent ϵ as an element in $\mathcal{O}_B \otimes \mathbf{Z}_p$. In view of the moduli interpretation, the p-divisible group $\mathcal{A}[p^m]$ has a natural structure of an $\mathcal{O}_B \otimes \mathbf{Z}_p$-module for a positive integer m. We thus have a natural action of the idempotent ϵ on the p-divisible group $\mathcal{A}[p^m]$. The action plays a key role in the article.

Heuristically, the idempotent often reduces the complexity arising from the non-commutative nature of the quaternion algebra to a commutative one. Such situations occasionally arise in the article.

Remark. An analogous idempotent arises in the case of an unitary model of a quaternionic Shimura variety over a totally real field (*cf.* [18] and [10]).

2.4. **CM points.** In this subsection, we introduce notation regarding CM points on the Shimura variety.

Let the notation and hypotheses be as in §2.2. Let b be a positive integer prime to N and recall that $\mathcal{O}_b$ denote the order $\mathbf{Z} + b\mathcal{O}$. We also recall that $\mathrm{Pic}(\mathcal{O}_b)$ denotes the corresponding ring

class group of conductor b.

Associated to an ideal class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_b)$, we have the corresponding CM point $x(\mathfrak{a})$ on the Shimura variety $Sh^{(p)}$ (*cf.* [6, §2.4] and [36, §1.1]). Moreover, these CM points are Galois conjugates (*cf.* [6, §2.4]).

2.5. **Igusa tower.** In this subsection, we briefly recall the notion of p-ordinary Igusa tower over the Shimura variety. We follow [24, Ch. 8] and [25, §1].

Let the notation and hypotheses be as in §2.2. Recall that p is a prime such that $p \nmid N$. Let $\mathcal{W}$ be the strict Henselisation inside $\overline{\mathbf{Q}}$ of the local ring of $\mathbf{Z}_{(p)}$ corresponding to ι_p. Let $\mathbb{F}$ be the residue field of $\mathcal{W}$. Note that $\mathbb{F}$ is an algebraic closure of $\mathbb{F}_p$.

Let $Sh^{(p)}_{/\mathcal{W}} = Sh^{(p)} \times_{\mathbf{Z}[\frac{1}{N}]} \mathcal{W}$ and $Sh^{(p)}_{/\mathbb{F}} = Sh^{(p)}_{/\mathcal{W}} \times_{\mathcal{W}} \mathbb{F}$.

From now, let Sh denote $Sh^{(p)}_{/\mathbb{F}}$. Recall that $\mathcal{A}$ denotes the universal abelian surface over Sh.

Let Sh^{ord} be the subscheme of Sh i.e. the locus on which the Hasse-invariant does not vanish. It is an open dense subscheme. Over Sh^{ord}, the connected part $\mathcal{A}[p^m]^\circ$ of $\mathcal{A}[p^m]$ is étale-locally isomorphic to $\mu_{p^m} \otimes_{\mathbf{Z}_p} \mathcal{O}_B^\circ$ as an $\mathcal{O}_{B,p}$-module. Here $\mathcal{O}_{B,p} = \mathcal{O}_B \otimes \mathbf{Z}_p$ and $\mathcal{O}_B^\circ$ is an $\mathcal{O}_{B,p}$-direct summand of $\mathcal{O}_{B,p}$. Moreover, $\mathcal{O}_B^\circ$ is a maximally isotropic subspace of $\mathcal{O}_{B,p}$ with respect to the bilinear form associated to the quadratic form arising from the reduced norm such that $\mathrm{rank}_{\mathbf{Z}_p} \mathcal{O}_B^\circ = 2$. Let $\mathcal{O}_B^{\acute{e}t}$ be the orthogonal complement of $\mathcal{O}_B^\circ$. For an explicit description of both, we refer to the end of §3.1.

We now define the Igusa tower. For $m \in \mathbb{N}$, the mth-layer of the Igusa tower over Sh^{ord} is defined by

$$(2.3) \qquad Ig_m = \underline{\mathrm{Isom}}_{\mathcal{O}_{B,p}}(\mu_{p^m} \otimes_{\mathbf{Z}_p} \mathcal{O}_B^\circ, \mathcal{A}[p^m]^\circ).$$

Note that the projection $\pi_m : Ig_m \to Sh^{ord}$ is finite and étale.

In view of the description of the moduli functor $\mathcal{F}^{(p)}$, we have the following isomorphism

$$(2.4) \qquad Ig_m \simeq \underline{\mathrm{Isom}}_{\epsilon \mathcal{O}_{B,p}}(\mu_{p^m} \otimes_{\mathbf{Z}_p} \epsilon \mathcal{O}_B^\circ, \epsilon(\mathcal{A}[p^m]^\circ)).$$

The Cartier duality and the polarisation $\bar{\lambda}_x$ induces an isomorphism

$$(2.5) \qquad Ig_m \simeq \underline{\mathrm{Isom}}_{\mathcal{O}_{B,p}}(\mathcal{O}_B^{\acute{e}t}/p^m \mathcal{O}_B^{\acute{e}t}, \mathcal{A}[p^m]^{\acute{e}t}) \simeq \underline{\mathrm{Isom}}_{\epsilon \mathcal{O}_{B,p}}(\epsilon(\mathcal{O}_B^{\acute{e}t}/p^m \mathcal{O}_B^{\acute{e}t}), \epsilon(\mathcal{A}[p^m]^{\acute{e}t})).$$

The full Igusa tower over Sh^{ord} is defined by

$$(2.6) \qquad Ig = \varprojlim Ig_m = \underline{\mathrm{Isom}}_{\mathcal{O}_{B,p}}(\mu_{p^\infty} \otimes_{\mathbf{Z}_p} \mathcal{O}_B^\circ, \mathcal{A}[p^\infty]^\circ) \simeq \underline{\mathrm{Isom}}_{\epsilon \mathcal{O}_{B,p}}(\mu_{p^\infty} \otimes_{\mathbf{Z}_p} \epsilon \mathcal{O}_B^\circ, \epsilon(\mathcal{A}[p^\infty]^\circ)).$$

In view of (2.5) and (2.6), it suffices to study the projection by ϵ of the p-divisible group.

(Et) Note that the projection $\pi : Ig \to Sh^{ord}$ is étale.

Let x be a closed ordinary point in Sh and A_x the corresponding abelian surface. We have the following description of the p^∞-level structure on $A_x[p^\infty]$.

(PL) Let η_p° be a p^∞-level structure on $A_x[p^\infty]^\circ$. It is given by an $\mathcal{O}_{B,p}$-module isomorphisms $\eta_p^\circ : \mu_{p^\infty} \otimes_{\mathbf{Z}_p} \mathcal{O}_B^\circ \simeq A_x[p^\infty]^\circ$. The Cartier duality and the polarisation $\bar{\lambda}_x$ induces an isomorphism $\eta_p^{\acute{e}t} : O_{B,p}^{\acute{e}t} \simeq A_x[p^\infty]^{\acute{e}t}$.

Let V be an irreducible component of Sh and V^{ord} the intersection $V \cap Sh^{ord}$. Let I be the inverse image of V^{ord} under π. In [24, Ch.8] and [25], it has been shown that

(Ir) I is an irreducible component of Ig.

2.6. Mod p modular forms.

In this subsection, we briefly recall the notion of mod p modular forms on an irreducible component of the Shimura variety.

Let V and I be as in §2.5. Let C be an $\mathbb{F}$-algebra. The space of mod p modular forms on V over C is defined by

$$(2.7) \qquad\qquad M(V,C) = H^0(I_{/C}, \mathcal{O}_{I_{/C}})$$

for $I_{/C} := I \times_{\mathbb{F}} C$.

In view of §2.2 and §2.5, we have the following geometric interpretation of mod p modular forms.

A mod p modular form is a function f of isomorphism classes of $\tilde{x} = (x, \eta_p^\circ)_{/C'}$ where C' is a C-algebra, $x = (A, \iota, \bar{\lambda}, \eta^{(p)})_{/C'} \in \mathcal{F}^{(p)}(C')$ and $\eta_p^\circ : \mu_{p^\infty} \otimes_{\mathbf{Z}_p} \mathcal{O}_B^\circ \simeq A[p^\infty]^\circ$ is an $\mathcal{O}_{B,p}$-linear isomorphism, such that the following conditions are satisfied.

(G1) $f(\tilde{x}) \in C'$.
(G2) If $\tilde{x} \simeq \tilde{x}'$, then $f(\tilde{x}) = f(\tilde{x}')$. Here $\tilde{x} \simeq \tilde{x}'$ denotes that $x \simeq x'$ and the corresponding isomorphism between A and A' induces an isomorphism between η_p° and $\eta_p'^\circ$.
(G3) $f(\tilde{x} \times_{C'} C'') = h(f(\tilde{x}))$ for any C-algebra homomorphism $h : C' \to C''$.

We have an analogous notion of p-adic (resp. classical) modular forms (resp., *cf.* [24, Ch. 8] and [6, §3.1]). In the article, we often regard classical modular forms as p-adic modular forms in the usual way.

3. Serre-Tate deformation space

In this section, we describe certain aspects of the Serre-Tate deformation space of the Shimura variety. In §3.1-3.2, we describe generalities regarding the Serre-Tate deformation theory of an ordinary closed point on the Shimura variety. In §3.3, we consider certain Hecke operators and determine their action on the Serre-Tate expansion of classical modular forms around CM points on the Shimura variety.

The deformation theory plays a foundational role in Chai's theory of Hecke stable subvarieties of a Shimura variety. This section thus plays a key role in the article.

3.1. **Serre-Tate deformation theory.** In this subsection, we briefly recall Serre-Tate deformation theory of an ordinary closed point on the Shimura variety. We follow [24, §8.2], [26, §2] and [27, §1].

Let the notation and assumptions be as in §2.5. In particular, Sh denotes the Shimura variety of level prime to $N^- p$ and Sh^{ord} the p-ordinary locus. Let x be a closed point in Sh^{ord} carrying $(A_x, \iota_x, \bar{\lambda}_x, \eta_x^{(p)})_{/\mathbb{F}}$. Let V be the irreducible component of Sh containing x.

Let CL_W be the category of complete local W-algebras with residue field $\mathbb{F}$. Let $\mathcal{D}_{/W}$ be the fiber category over CL_W of deformations of $x_{/\mathbb{F}}$ defined as follows. Let $R \in CL_W$. The objects of $\mathcal{D}_{/W}$ over R consist of $x'^* = (x', \iota_{x'})$, where $x' \in \mathcal{F}^{(p)}(R)$ and $\iota_{x'} : x' \times_R \mathbb{F} \simeq x$. Let x'^* and x''^* be in $\mathcal{D}_{/W}$ over R. By definition, a morphism ϕ between x'^* and x''^* is a morphism (still denoted by) ϕ between x' and x'' satisfying [24, (7.3)] and the following condition.

(M) Let ϕ_0 be the special fiber of ϕ. The automorphism $\iota_{x''} \circ \phi_0 \circ \iota_{x'}^{-1}$ of x equals the identity.

Let $\widehat{\mathcal{F}}_x$ be the deformation functor given by

$$\widehat{\mathcal{F}}_x : CL_{/W} \to SETS$$

(3.1)
$$R \mapsto \{x'^*_{/R} \in \mathcal{D}\}/ \simeq .$$

The notation $\simeq$ denotes up to an isomorphism.

As $R \in CL_W$, by definition R is a projective limit of local W-algebras with nilpotent maximal ideal. We can (and do) suppose that R is a local Artinian W-algebra with the nilpotent maximal ideal m_R. Let $x'^*_{/R} \in \mathcal{D}$ and A denote $A_{x'}$. By Drinfeld's theorem (*cf.* [24, §8.2.1]), $A[p^\infty]^\circ(R)$ is killed by p^{n_0} for sufficiently large n_0. Let $y \in A(\mathbb{F})$ and $\tilde{y} \in A(R)$ such that $\tilde{y}_0 = y$ for $\tilde{y}_0$ being the special fiber of $\tilde{y}$. As $A_{/R}$ is smooth, such a lift always exists. By definition, $\tilde{y}$ is determined modulo $\ker(A(R) \mapsto A(\mathbb{F})) = A[p^\infty]^\circ(R)$. It thus follows that for $n \geq n_0$, "p^n"$y_0 := p^n \tilde{y}$ is well defined. From now, we suppose that $n \geq n_0$. If $y \in A[p^n](\mathbb{F})$, then "p^n"$y \in A[p^\infty]^\circ(R)$ by definition. We now consider the ϵ-components (*cf.* §2.3). Recall that it suffices to consider the ϵ-components in view of the isomorphisms (2.5) and (2.6).

We thus have a homomorphism

(3.2)
$$\text{``}p^n\text{''} : \epsilon A[p^n](\mathbb{F}) \to \epsilon A[p^\infty]^\circ(R).$$

We also have the commutative diagram

$$
\begin{array}{ccccc}
\epsilon A[p^{n+1}]^{\acute{e}t}(R) & \overset{\simeq}{\to} & \epsilon A[p^{n+1}](\mathbb{F}) & \xrightarrow{\text{``}p^{n+1}\text{''}} & \epsilon A[p^\infty]^\circ(R) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle =} \\
\epsilon A[p^n]^{\acute{e}t}(R) & \overset{\simeq}{\to} & \epsilon A[p^n](\mathbb{F}) & \xrightarrow{\text{``}p^n\text{''}} & \epsilon A[p^\infty]^\circ(R).
\end{array}
$$

Passing to the projective limit, this gives rise to a homomorphism

(3.3)
$$\text{``}p^\infty\text{''} : \epsilon A[p^\infty](\mathbb{F}) \to \epsilon A[p^\infty]^\circ(R).$$

(CC) Let $y = \varprojlim y_n \in \epsilon A[p^\infty](\mathbb{F}) \simeq \epsilon A_x[p^\infty]^{\acute{e}t}$, where $y_n \in \epsilon A[p^n](\mathbb{F})$ and the later isomorphism is

induced by $\iota_{x'}$. Let $q_{n,p}(y_n) = \text{``}p^n\text{''}y_n$ and $q_p(y) = \lim q_{n,p}(y_n)$. By definition,

$$q_p(y) \in \epsilon A[p^\infty]^\circ(R) \simeq \text{Hom}(\epsilon^t A_x[p^\infty]^{\text{ét}}, \widehat{\mathbb{G}}_m(R)).$$

Let $q_{A,p}$ be the pairing given by

$$q_{A,p} : \epsilon A_x[p^\infty]^{\text{ét}} \times \epsilon^t A_x[p^\infty]^{\text{ét}} \to \widehat{\mathbb{G}}_m(R)$$

(3.4)
$$q_{A,p}(y,z) = q_p(y)(z).$$

We have the following fundamental result.

Theorem 3.1 (Serre-Tate). *Let the notation be as above.*
(1). There exists a canonical isomorphism

(3.5)
$$\widehat{\mathcal{F}}_x(R) \simeq \text{Hom}_{\mathbf{Z}_p}(\epsilon A_x[p^\infty]^{\text{ét}} \times \epsilon^t A_x[p^\infty]^{\text{ét}}, \widehat{\mathbb{G}}_m(R))$$

given by $x'^* \mapsto q_{A_{x'},p}$.
(2). The deformation functor $\widehat{\mathcal{F}}_x$ is represented by the formal scheme $\widehat{S}_{/W} := \text{Spf}(\widehat{\mathcal{O}}_{V,x})$. A p^∞-level structure as in (PL), gives rise to a canonical isomorphism of the deformation space $\widehat{S}_{/W}$ with the formal torus $\widehat{\mathbb{G}}_m$ (cf. [27, Prop. 1.2]).

PROOF. We first prove (1). It follows from an argument similar to the proof of [24, Thm. 8.9]. We sketch the details for convenience.

A fundamental result of Serre and Tate states that deforming x over CL_W is equivalent to deform the corresponding Barsotti-Tate $\mathcal{O}_{B,p}$-module $A_x[p^\infty]_{/\mathbb{F}}$ over CL_W (cf. [35, §1] and [24, §8.2.2]). In view of the isomorphisms (2.5) and (2.6), deforming the Barsotti-Tate $\mathcal{O}_{B,p}$-module $A_x[p^\infty]_{/\mathbb{F}}$ is equivalent to deform the Barsotti-Tate $\epsilon\mathcal{O}_{B,p}$-module $\epsilon A_x[p^\infty]_{/\mathbb{F}}$.

Let R be as above. Consider the sheafification $\underline{\text{Hom}}_{R_{fppf}}(\epsilon A[p^n]^\circ, \epsilon A[p^n]^{\text{ét}})$ (resp. $\underline{Ext}^1_{R_{fppf}}(\epsilon A[p^n]^\circ, \epsilon A[p^n]^{\text{ét}})$) of the presheaf $U \mapsto \text{Hom}_U(\epsilon A[p^n]^\circ_{/U}, \epsilon A[p^n]^{\text{ét}}_{/U})$ (resp. $U \mapsto Ext^1_U(\epsilon A[p^n]^\circ_{/U}, \epsilon A[p^n]^{\text{ét}}_{/U})$). Recall that a connected-étale extension $\epsilon A[p^n]^\circ \to X \twoheadrightarrow \epsilon A[p^n]^{\text{ét}}$ in the category of finite flat $\mathbf{Z}/p^n\mathbf{Z}$-modules over R splits over an *fppf*-extension R'/R. We thus have

$$\underline{Ext}^1_{R_{fppf}}(\epsilon A[p^n]^\circ, \epsilon A[p^n]^{\text{ét}}) = 0$$

and a splitting

$$X = \epsilon A[p^n]^\circ \oplus \epsilon A[p^n]^{\text{ét}}.$$

Choice of a section in the splitting gives rise to a homomorphism $\phi_{R'} \in \text{Hom}_{R'}(\epsilon A[p^n]^\circ_{/R'}, \epsilon A[p^n]^{\text{ét}}_{/R'})$. By construction, $R' \mapsto \phi_{R'}$ satisfies the descent datum. We thus have a morphism

$$Ext^1_{R_{fppf}}(\epsilon A[p^n]^\circ, \epsilon A[p^n]^{\text{ét}}) \to H^1(R_{fppf}, \underline{\text{Hom}}_{R_{fppf}}(\epsilon A[p^n]^\circ, \epsilon A[p^n]^{\text{ét}})).$$

By an *fppf*-descent, this is in fact an isomorphism. We conclude that

$$Ext^1_{R_{fppf}}(\epsilon A[p^\infty]^\circ, \epsilon A[p^\infty]^{\text{ét}}) \simeq \varprojlim_n Ext^1_{R_{fppf}}(\epsilon A[p^n]^\circ, \epsilon A[p^n]^{\text{ét}}) \simeq \text{Hom}_{\mathbf{Z}_p}(\epsilon A_x[p^\infty]^{\text{ét}} \times \epsilon^t A_x[p^\infty]^{\text{ét}}, \widehat{\mathbb{G}}_m(R)).$$

The last isomorphism follows from the additional fact that $\widehat{\mathbb{G}}_m \simeq \varprojlim_n R^1\pi_*\mu_{p^n} \simeq \widehat{\mathbb{G}}_m$. Here $\pi : R_{fppf} \to R_{\text{ét}}$ is the projection for the small étale site $R_{\text{ét}}$.

We conclude that the deformation functor $\widehat{\mathcal{F}}_x$ is represented by $\text{Hom}_{\mathbf{Z}_p}(\epsilon A_x[p^\infty]^{\text{ét}} \times \epsilon^t A_x[p^\infty]^{\text{ét}}, \widehat{\mathbb{G}}_m(R))$

and this finishes the proof of (1).

We now prove (2). The first part of (2) follows from general deformation theory. The polarisation λ_x induces a canonical $\mathcal{O}_B$-linear isomorphism $A_x \simeq^t A_x$. For a given choice of p^∞-level structure $\eta_p^{\acute{e}t}$ as in (PL), we have a canonical isomorphism $\mathrm{Hom}_{\mathbf{Z}_p}(\epsilon A_x[p^\infty]^{\acute{e}t} \times \epsilon A_x[p^\infty]^{\acute{e}t}, \widehat{\mathbb{G}}_m) \simeq \widehat{\mathbb{G}}_m$. This finishes the proof.

$\square$

Let x_{ST} be the universal deformation. A p^∞-level structure η_p° of x gives rise to a canonical p^∞-level structure $\eta_{p,ST}^\circ$ of the universal deformation x_{ST}.

We now recall a few definitions.

Definition 3.2. Recall that a p^∞-level structure as in (PL), gives rise to a canonical isomorphism of the deformation space $\widehat{S}_{/W}$ with the formal torus $\widehat{\mathbb{G}}_m$ (*cf.* part (2) of Theorem 3.1). Under this identification, let t be the co-ordinate of the deformation space $\widehat{S}_{/W}$ for t being the usual co-ordinate of $\widehat{\mathbb{G}}_m$. We call t the Serre-Tate co-ordinate of the deformation space $\widehat{S}_{/W}$.

Definition 3.3. The deformation corresponding to the identity element in the deformation space is said to be the canonical lift of x.

We have $\widehat{S} = \mathrm{Spf}(W\widehat{[X(S)]})$ for $S = \mathbb{G}_m$ and $W\widehat{[X(S)]}$ being the completion along the augmentation ideal. Here $X(S)$ is the character group of S. Note that $W[X(S)]$ is the ring consisting of formal finite sums $\sum_{\xi \in \mathbf{Z}} a(\xi) t^\xi$ for $a(\xi) \in W$ and t being the usual co-ordinate of $\mathbb{G}_m$.

Definition 3.4. Let f be a mod p modular form over $\mathbb{F}$ (*cf.* §2.6). We call the evaluation $f((x_{ST}, \epsilon\eta_{p,ST}^\circ)) \in \mathbb{F}\widehat{[X(S)]}$ as the t-expansion of f around x.

We have the following t-expansion principle.

(t-expansion principle) The t-expansion of f around x determines f uniquely (*cf.* (Ir)).

We have an analogous t-expansion principle for p-adic modular forms (*cf.* [24, Ch. 8]).

We end this subsection with a description of the modules $\mathcal{O}_B^\circ$ and $\mathcal{O}_B^{\acute{e}t}$ (*cf.* §2.5).

We follow [25, pp. 9-10]. We fix an identification of $\mathcal{O}_{B,p}$ with $\mathbf{Z}_p^4 = (\mathbf{Z}_p^2)^2$. Let $\mathcal{O}_{B,p}$ act on itself by the left multiplication on $\mathbf{Z}_p^2$. Let x be a closed point in Sh^{ord} carrying $(A_x, \iota_x, \bar{\lambda}_x, \eta_x^{(p)})_{/\mathbb{F}}$. Let $\tilde{x} = (A_x, \iota_x, \bar{\lambda}_x, \eta_x^{(p)})_{/W}$ be the canonical lift of x. As $A_{x/W}$ has CM, it descends to $A_{x/\mathcal{W}}$ (*cf.* [45]). Let $A_{x/\mathbf{C}} = A_x \times_{\mathcal{W}, \iota_\infty} \mathbf{C}$. Note that $\mathcal{T}(A_x)_{/\overline{E}} \simeq H_1(A_x(\mathbf{C}), \mathbf{A}_{\mathbf{Q},f}) = H_1(A_x(\mathbf{C}), \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{A}_{\mathbf{Q},f}$. Let $\mathcal{T}_p A_{x/\mathbf{C}}$ be the p-adic Tate-module of $A_{x/\mathbf{C}}$. We choose an identification of $\mathcal{T}_p A_{x/\mathbf{C}}$ with $\mathcal{O}_{B,p} = H_1(A_x(\mathbf{C}), \mathbf{Z}_p)$. The connected-étale exact sequence $A_x[p^\infty]^\circ \hookrightarrow A_x[p^\infty] \twoheadrightarrow A_x[p^\infty]^{\acute{e}t}$ of the p-divisible group $A_x[p^\infty]_{/\mathbb{F}}$ induces an exact sequence $\mathcal{O}_B^\circ \hookrightarrow \mathcal{O}_{B,p} \twoheadrightarrow \mathcal{O}_B^{\acute{e}t}$ of $\mathcal{O}_{B,p}$-modules. As $A_{x/\mathcal{W}}$ is the

canonical lift, it follows that $\mathcal{O}_{B,p}$ splits as $\mathcal{O}_B^\circ \oplus \mathcal{O}_B^{\text{ét}}$ such that the lift of the Frobenius has unit eigenvalues on $\mathcal{O}_B^{\text{ét}}$ and non-unit eigenvalues on $\mathcal{O}_B^\circ$.

3.2. **Reciprocity law.** In this subsection, we describe the action of the local algebraic stabiliser of an ordinary closed point on the Serre-Tate co-ordinates of the deformation space. This can be considered as an infinitesimal analogue of Shimura's reciprocity law.

Let the notation and hypotheses be as in §3.1. In particular, Sh denotes the Shimura variety of level prime to $N^- p$. Let $g \in G(\mathbf{Z}_{(p)})$ acts on Sh through the right multiplication on the prime to $N^- p$ level structure i.e.,

$$(A, \iota, \bar{\lambda}, \eta^{(p)})_{/S} \mapsto (A, \iota, \bar{\lambda}, \eta^{(p)} \circ g)_{/S}$$

(*cf.* §2.2).

Recall that x is a closed ordinary point in Sh with a p^∞-level structure η_p^{ord}. We suppose that x has CM by K and the existence of an embedding $\iota_x : \mathcal{O} \hookrightarrow \text{End}(A)$. Let $H_x(\mathbf{Z}_{(p)})$ be the stabiliser of x in $G(\mathbf{Z}_{(p)})$. It is explicitly given by

$$(3.6) \qquad\qquad H_x(\mathbf{Z}_{(p)}) = (\text{Res}_{\mathcal{O}_{(p)}/\mathbf{Z}_{(p)}} \mathbb{G}_m)(\mathbf{Z}_{(p)}) = \mathcal{O}_{(p)}^\times$$

for $\mathcal{O}_{(p)} = \mathcal{O} \otimes \mathbf{Z}_{(p)}$ (*cf.* [26, §3.2] and [45]). We call $H_x(\mathbf{Z}_{(p)})$ the local algebraic stabiliser of x.

Recall that c denotes the complex conjugation of K.

Let $\alpha \in H_x(\mathbf{Z}_{(p)})$. Let $R \in CL_W$, $x' \in \widehat{\mathcal{F}}_x(R)$ and $A = A_{x'}$. We have the following

$$
\begin{array}{ccccc}
\text{Hom}(\epsilon A_x[p^\infty]^{\text{ét}}, \hat{\mathbb{G}}_m(R)) & \hookrightarrow & \epsilon A[p^n] & \twoheadrightarrow & \epsilon A_x[p^n]^{\text{ét}}(R) \\
\big\uparrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \text{id}} & & \big\downarrow{\scriptstyle \alpha^{-c}} \\
\text{Hom}(\epsilon A_x[p^\infty]^{\text{ét}}, \hat{\mathbb{G}}_m(R)) & \hookrightarrow & \epsilon A[p^n] & \twoheadrightarrow & \epsilon A_x[p^n]^{\text{ét}}(R)
\end{array}
$$

commutative diagram.

Let $u = \varprojlim u_n \in \epsilon A_x[p^\infty]^{\text{ét}}$ and $q_p(u) \in \epsilon A_x[p^\infty]^\circ(R) \simeq \text{Hom}(\epsilon A_x[p^\infty]^{\text{ét}}, \hat{\mathbb{G}}_m(R))$ be as in (CC). We recall that the last isomorphism is induced by the Cartier duality composed with the polarisation λ_x. Thus, if α is prime to p, then it acts on $q_p(u)$ by

$$q_p(u) \mapsto \varprojlim \alpha(\text{"}p^n\text{"} \alpha^{-c_x}(u_n)) = q_p(\alpha)^{\alpha^{1-c_x}}.$$

In other words, α acts on q_p by $q_p \mapsto q_p^{\alpha^{1-c}}$. Fix a p^∞-level structure as in (PL). This gives rise to the Serre-Tate co-ordinate t of the deformation space $\widehat{S}_{/W}$ (cf. (CC)).

In view of the above discussion and Theorem 3.1, we have the following reciprocity law.

Lemma 3.5. *Let the notation and the assumptions be as above. If $\alpha \in H_x(\mathbf{Z}_{(p)})$ is prime to p, then it acts on the Serre-Tate co-ordinate t by $t \mapsto t^{\alpha^{1-c}}$.*

The above simple lemma plays a key role in the proof of a linear independence of mod p modular forms (*cf.* §4).

3.3. **Hecke operators.** In this subsection, we describe certain Hecke operators on the space of classical modular forms. In the case of modular curves, such operators are considered in [28, §1.3.5].

Let the notation and hypotheses be as in §3.1. Let $(\zeta_{p^n} = \exp(2\pi i/p^n))_n \in \overline{\mathbf{Q}}$ be a compatible system of p-power roots of unity. Via ι_p, we regard it as a compatible system in $\mathbf{C}_p$. Let $W_n = W[\mu_{p^n}]$.

Let g be a classical modular form on a quotient Shimura curve Sh_K of the Shimura variety Sh over $\mathcal{W}$ (*cf.* §2.1). Via ι_∞, we regard it as a modular form over $\mathbf{C}$. Let z denote the complex variable of the complex Shimura curve or that of $\mathfrak{H}$ for $\mathfrak{H}$ being the upper half complex plane.

Let $\phi : \mathbf{Z}/p^r\mathbf{Z} \to \mathcal{W}$ be an arbitrary function with the normalised Fourier transform ϕ^* given by

$$(3.7) \qquad \phi^*(y) = \frac{1}{p^{r/2}} \sum_{u \in \mathbf{Z}/p^r\mathbf{Z}} \phi(u) \exp(yu/p^r)$$

for $y \in \mathbf{Z}/p^r\mathbf{Z}$.

Let $g|\phi$ be the classical modular form given by

$$(3.8) \qquad g|\phi(z) = \sum_{u \in \mathbf{Z}/p^r\mathbf{Z}} \phi^*(-u)g(z + u/p^r).$$

We now regard the above classical modular forms as p-adic. The action of the Hecke operator on the t-expansion around a CM point (*cf.* Definition 3.4) is the following.

Proposition 3.6. *Let the notation be as above. Let $g(t) = \sum_{\xi \in \mathbf{Z}_{\geq 0}} a(\xi, g)t^\xi$ be the t-expansion of g around x. We have*

$$g|\phi(t) = \sum_{\xi \in \mathbf{Z}_{\geq 0}} \phi(\xi)a(\xi, g)t^\xi.$$

PROOF.   Let $u \in \mathbf{Z}$ and $\alpha(u/p^r) \in G(\mathbf{A}_{\mathbf{Q},f})$ such that

$$\alpha(u/p^r)_p = \begin{bmatrix} 1 & u/p^m \\ 0 & 1 \end{bmatrix}$$

and $\alpha(u/p^r)_l = 1$, for $l \neq p$.

We have

$$g|\phi = \sum_{u \in \mathbf{Z}/p^r\mathbf{Z}} \phi^*(-u)g|\alpha(u/p^r),$$

as $\alpha(u/p^r)$ acts on $\mathfrak{H}$ by $z \mapsto z + u/p^r$. Here $\mathfrak{H}$ denotes the complex upper half plane as before.

The isogeny action of $\alpha(u/p^r)$ on the Igusa tower $\pi : Ig \to Sh_{/W_r}$ preserves the deformation space $\widehat{S}$ and induces $t \mapsto \zeta_{p^r}t^u$ (*cf.* [6, Lem. 4.14]).

In view of the Fourier inversion formula, this finishes the proof.

$\square$

The above Hecke operators naturally arise during the determination of the μ-invariant of a class of anticyclotomic Rankin-Selberg p-adic L-functions (*cf.* Lemma 5.4 and proof of Theorem 5.7). The above proposition accordingly helpful in the determination of the μ-invariants.

4. Linear independence

In this section, we consider a linear independence of mod p modular forms based on Chai's theory of Hecke stable subvarieties of a Shimura variety. In §4.1, we describe the formulation. In §4.2-4.6, we prove the independence.

4.1. **Formulation.**
In this subsection, we give a formulation of the linear independence of mod p modular forms.

Let the notation and hypotheses be as in §2.5. Recall that x is a closed ordinary point in V with p^∞-level structure η_p°. This gives rise to a closed point $\tilde{x} = (x, \eta_p^\circ)$ in the Igusa tower I over x. Recall that we have a canonical isomorphism $\widehat{\mathcal{O}}_{V,x} \simeq \widehat{\mathcal{O}}_{I,\tilde{x}}$ (*cf.* (Et)). Thus, we have a natural action of the p-adic stabiliser $H_x(\mathbf{Z}_p)$ on $\widehat{\mathcal{O}}_{I,\tilde{x}}$ (*cf.* §3.2).

Our formulation of the independence is the following.

Theorem 4.1 (Linear independence)**.** *Let x be a closed ordinary point on the Shimura variety with the local stabiliser H_x. For $1 \le i \le n$, let $a_i \in H_x(\mathbf{Z}_p)$ such that $a_i a_j^{-1} \notin H_x(\mathbf{Z}_{(p)})$ for all $i \neq j$ (cf. §3.2). Let $f_1, ..., f_n$ be non-constant mod p modular forms on V. Then, $(a_i \circ f_i)_i$ are linearly independent in the Serre-Tate deformation space $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$.*

In general, note that $a_i \circ f_i$ is not a mod p modular form.

We actually prove the algebraic independence of $(a_i \circ f_i)_i$. The theorem is an analogue of Ax-Lindemann-Weierstrass conjecture for the mod p reduction of the Shimura variety.

The theorem is proven in §4.6.

4.2. **Locally stable subvarieties.**
In this subsection, we describe the notion and the results regarding locally stable subvarieties of a self-product of the Shimura variety.

Let the notation and hypotheses be as in §2.5. Let n be a positive integer. In this subsection, any tensor product is taken n times.

Let Sh be as before and V an irreducible component of Sh.

Definition 4.2. A subvariety Y of V^n is said to be locally stable if there exists a closed point $y^n = (y, ..., y) \in Y$ such that Y is stable under the diagonal action of the local algebraic stabilser $H_y(\mathbf{Z}_{(p)})$.

While considering the problem of linear independence, this type of subvariety arises as follows.

We consider an $\mathbb{F}$-algebra homomorphism

(4.1) $$\phi_I : \mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}} \to \widehat{\mathcal{O}}_{I,\tilde{x}}$$

given by

(4.2) $$f_1 \otimes ... \otimes f_n \mapsto \prod_{i=1}^{n} a_i \circ f_i.$$

As we are interested in the independence of $(a_i \circ f_i)_i$, we consider $\mathfrak{b}_I := ker(\phi_I)$.

Similarly, we consider an $\mathbb{F}$-algebra homomorphism

(4.3) $$\phi = \phi_V : \mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x} \to \widehat{\mathcal{O}}_{V,x}$$

given by

(4.4) $$h_1 \otimes ... \otimes h_n \mapsto \prod_{i=1}^{n} a_i \circ h_i.$$

The following simple observation is crucial in what follows.

(EQ) The homomorphism ϕ is equivariant with the $H_x(\mathbf{Z}_{(p)})$-action.

Let $\mathfrak{b} = ker(\phi_V)$.

Lemma 4.3. *Let the notation be as above. We have* $\mathfrak{b}_I = 0$ *if and only if* $\mathfrak{b} = 0$.

PROOF.   In view of (Et), we have an étale morphism $\pi^n : \mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x} \to \mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}}$. As $\mathfrak{b}_I$ is the unique prime ideal of $\mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}}$ over $\mathfrak{b}$, this finishes the proof.

$\square$

In view of (EQ), it follows that $\mathfrak{b}$ is a prime ideal of $\mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x}$ stable under the diagonal action of $H_x(\mathbf{Z}_{(p)})$. Let X be the Zariski closure of $\mathrm{Spec}\,(\mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x}/\mathfrak{b})$ in V^n. Thus, X is a closed irreducible subscheme of V^n containing x^n stable under the diagonal action of $H_x(\mathbf{Z}_{(p)})$. In particular, X is a closed irreducible locally stable subvariety of V^n.

We now axiomatise the above example.

(LS1) Let $S_{/\mathbb{F}}$ be $\mathrm{Spec}\,(\mathcal{O}_{V,x})$.
(LS2) Let $\mathfrak{b}$ be a prime ideal of $\mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x}$ stable under the diagonal action of $H_x(\mathbf{Z}_{(p)})$ and $\mathcal{X}_{/\mathbb{F}} = \mathrm{Spec}\,(\mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x}/\mathfrak{b})$.
(LS3) Let X be the Zariski closure of $\mathcal{X}$ in V^n.

Note that X is positive dimensional as $\mathfrak{b}$ is a prime ideal.

A natural class of locally stable subvarieties is the following.

Definition 4.4. Let l be a positive integer. For $1 \leq i \leq l$, let $\alpha_i \in H_x(\mathbf{Z}_{(p)})$. The skewed diagonal $\Delta_{\alpha_1,\dots,\alpha_l}$ of V^l is defined by $\Delta_{\alpha_1,\dots,\alpha_l} = \big\{(\alpha_1(y), \dots, \alpha_l(y)) : y \in V\big\}$.

The global structure of a class of locally stable subvarieties is given by the following result.

Theorem 4.5. *Let $n \geq 2$ and X be locally stable as in (LS1)-(LS3). Let S' be the product of the first $n-1$ factors of S^n and $S'' = S$ be the last factor of S^n. Suppose that the projections $\pi'_{\mathcal{X}} : \mathcal{X} \to S'$ and $\pi''_{\mathcal{X}} : \mathcal{X} \to S''$ are dominant. Then, X equals V^n or $V^{n-2} \times \Delta_{\alpha_1,\alpha_2}$ for some $\alpha_1, \alpha_2 \in H_x(\mathbf{Z}_{(p)})$ up to a permutation of the first $n-1$ factors.*

The theorem is an instance of Chai-Oort principle that a Hecke stable subvariety of a Shimura variety is a Shimura subvariety (*cf.* [19], [20] and [21]). The principle is an analogue of Andre-Oort conjecture for mod p Shimura varieties.

The theorem is proven in §4.3-§4.5.

In [26], an analogue of the theorem is proven in the Hilbert modular case. For the proof, we follow the same strategy as in [19], [20] and [26, §3]. We first consider the structure of $\mathrm{Spf}(\widehat{X}_{x^n})$ as a formal subscheme of $\mathrm{Spf}(\widehat{\mathcal{O}}_{V^n, x^n})$. Recall that the Serre-Tate deformation space $\mathrm{Spf}(\widehat{\mathcal{O}}_{V^n, x^n})$ has a natural structure of a formal torus. As $\mathrm{Spf}(\widehat{X}_{x^n})$ is a formal subscheme of $\mathrm{Spf}(\widehat{\mathcal{O}}_{V^n, x^n})$ stable under the diagonal action of the stabiliser $H_x(\mathbf{Z}_{(p)})$ and (pretending) X is smooth, it follows from Chai's local rigidity that $\mathrm{Spf}(\widehat{X}_x)$ is in fact a formal subtorus of $\mathrm{Spf}(\widehat{\mathcal{O}}_{V^n, x^n})$. In view of Theorem 3.1, we have a description of $\mathrm{Spf}(\widehat{\mathcal{O}}_{V^n, x^n})$ as a formal torus. Based on it, we obtain a description of the formal subtori of $\mathrm{Spf}(\widehat{\mathcal{O}}_{V^n, x^n})$ stable under the diagonal action of the stabiliser $H_x(\mathbf{Z}_{(p)})$. We thus obtain an explicit description of the possibilities for the structure of $\mathrm{Spf}(\widehat{X}_{x^n})$ as a formal torus. Based on de Jong's result on Tate conjecture in [22], we can eliminate all but one possibilities. Strictly speaking, the proof is a bit more involved as we cannot directly suppose that X is smooth. Instead, we apply the strategy to the normalisation of X and later show that the normalisation equals X itself.

4.3. **Local structure.** In this subsection, we consider the local structure of a class of locally stable subvarieties of a self-product of the Shimura curve.

Let the notation and hypotheses be as in §4.2. However, we do not suppose that the projection $\pi''_{\mathcal{X}}$ is dominant. Let $\Pi_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{X}$ be the normalisation of $\mathcal{X}$.

The following is an analogue of [26, Prop. 3.11] to our setting.

Proposition 4.6. *Let the notation and assumptions be as above. Suppose that the projection $\pi'_{\mathcal{X}} : \mathcal{X} \to S'$ is dominant. Then, the following holds.*

(1). The normalisation $\mathcal{Y}$ has only finitely many points y over x^n. For any such y, we have $\widehat{\mathcal{Y}}_y = \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} L_y$ for an $\mathbf{Z}_p$-direct summand L_y of $X_*(\widehat{S^n})$. Moreover, the isomorphism class of L_y as an $\mathbf{Z}_p$-module is independent of y.
(2). The normalisation $\mathcal{Y}$ is smooth over $\mathbb{F}$ and flat over S'.
(3). Either $\mathcal{X} = S^n$ or $\mathcal{X}$ is finite over S'. In the later case, the normalisation $\mathcal{Y}$ is finite flat over S'.
(4). If $\pi'_\mathcal{X} \circ \Pi_\mathcal{Y}$ induces a surjection of the tangent space at y onto S' at x^n for some y over x^n, then $\pi'_\mathcal{X} \circ \Pi_\mathcal{Y} : \mathcal{Y} \to S'$ is étale.

PROOF.   The proof is similar to the proof of Proposition 3.11 in [26]. Here we only prove (1).

Let $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}_{x^n}$. By the local stabiliser principle (cf. [21, Prop. 6.1]), $\widehat{\mathcal{X}}$ is a formal subscheme of $\widehat{S^n}$ stable under the diagonal action of the p-adic stabiliser $H_x(\mathbf{Z}_p)$. In view of Chai's local rigidity (cf. [20, §6]), it follows that $\widehat{\mathcal{X}} = \bigcup_{L \in I} \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} L$. Here L is an $\mathbf{Z}_p$-direct summand of $X_*(\widehat{S^n})$ (cf. Theorem 3.1) and the union is finite.

Note that the semigroup $\mathrm{End}_{SCH}(\mathcal{X})$ naturally acts on $\mathcal{Y}$. In particular, the stabiliser $H_x(\mathbf{Z}_{(p)})$ acts on $\mathcal{Y}$. As $\widehat{\mathcal{Y}}_y$ is integral, the points y are indexed by the irreducible components of $\widehat{\mathcal{X}}$. Let y_L be the point corresponding to L. As the morphism $\pi'_\mathcal{X} \circ \Pi_\mathcal{Y}$ is dominant, there exists at least one $L_0 \in I$ such that the image of L_0 is of finite index in $X_*(\widehat{S'})$.

To arrive at the desired conclusion, we consider a global argument as follows. Let $\Pi : Y \to X$ be the normalisation of X. As before, the local algebraic stabiliser $H_x(\mathbf{Z}_{(p)})$ acts on Y. Recall that the order $\mathcal{O}_B$ acts on X and hence on Θ_X. This action extends to Θ_Y. In view of the Kodaira-Spencer isomorphism for the universal abelian scheme $\mathcal{A}_{/Y}$ over Y and the identification $\widehat{Y}_{y_L} = \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} L$, it follows that $\mathrm{rank}_{\mathbf{Z}_p} L = \mathrm{rank}_{\mathcal{O}_Y} \Theta_Y$. This finishes the proof of (1).

$\square$

4.4. Global structure I.
In this subsection, we study global structure of a class of locally stable subvarieties of a 2-fold self-product of the Shimura curve.

Let the notation and hypotheses be as in §4.2 and $n = 2$. In view of Proposition 4.6, we conclude that either $\dim \mathcal{X} = \dim V$ or $X = V^2$. From now, we suppose that $\dim \mathcal{X} = \dim V$. In the rest of the subsection, we let $i = 1, 2$.

We need to show that X is a graph of an automorphism of V arising from $H_x(\mathbf{Z}_{(p)})$. In other words, we need to show that the projection $\pi_i : X \to V_i$ is an isomorphism. Here π_i is the projection to the ith-factor and we view $V_i = V$ as the ith factor of $V \times V$. Our strategy is the following.

Step 1. The morphism $\Pi_i : Y \to X \to V$ is étale over an open dense subscheme of V. This is proven based on Serre-Tate deformation theory and Proposition 4.6.

Step 2. Let $\mathcal{A}_i = \Pi_i^* \mathcal{A}$ and $\eta_i^{(p)} = \Pi_i^* \eta^{(p)}$. There exists an isogeny $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$. This is proven based

on de Jong's theorem in [22]. In particular, there exists $g \in G(\mathbf{A}_{\mathbf{Q},f}^{(p)})$ such that $\varphi \circ \eta_1^{(p)} = \eta_2^{(p)} \circ g$. We consider the fiber at x and conclude that $g \in H_x(\mathbf{Z}_{(p)})$. Finally, we deduce that $Y = X$ and $X = \Delta_{1,g}$.

We now describe each step. In view of Proposition 4.6 and our assumption $\dim \mathcal{X} = \dim V$, it follows that the morphism $\Pi_i : Y \to V$ is finite at any point $y \in Y$ above x^2. If Π_i is not étale at y, then $\Pi_{i,*}(X_*(\widehat{Y}_y)) \subset X_*(\widehat{S})$ is a $\mathbf{Z}_p$-submodule of finite index. Let $\alpha \in H_x(\mathbf{Z}_{(p)})$ such that $\alpha^{1-c} X_*(\widehat{S}) = \Pi_{i,*}(X_*(\widehat{Y}_y))$. The morphism $\alpha^{-1} \circ \Pi_i$ is étale. Thus, from now we suppose that

(E) the morphism $\Pi_i : Y \to V$ is étale finite at any point $y \in Y$ over x^2.

Let $V^{ét}$ be the maximal subscheme over which both Π_1 and Π_2 are étale. Let Π_i also denote the projection $\mathcal{Y} \to S$.

Lemma 4.7. *Let the notation and assumptions be as above. If the projections $\Pi_1 : \mathcal{Y} \to S$ and $\Pi_2 : \mathcal{Y} \to S$ are both étale, then $V^{ét}$ is an open dense subscheme of V^2 containing x^2 and stable under the diagonal action of the stabiliser $H_x(\mathbf{Z}_{(p)})$.*

PROOF.   The later part follows from the definition. The former part can be proven in a way similar to the proof of [26, Prop. 3.14]. In fact, the proof is simpler in our case as Sh is proper. $\qquad \square$

Let $Y^{ord} = Y \times_{V^2} (V^{ord})^2$. Let $y' \in Y^{ord}$ be a closed point above $(x_1, x_2) \in (V^{ord})^2$.

Definition 4.8. Y is said to be $\mathcal{O}_B$-linear at y' if $\Pi_1 \times \Pi_2$ embeds $\widehat{Y}_{y'}$ into $\widehat{V}_{x_1} \times \widehat{V}_{x_2}$ and the equation defining $\widehat{Y}_{y'}$ is given by $t_1^a = t_2^b$, where t_i is the Serre-Tate co-ordinate of $\widehat{V}_{x_i}$ (cf. §3.1) and $a, b \in \mathrm{End}_{\epsilon \mathcal{O}_{B,p}}(\widehat{S})$.

Let $Y^{lin} \subset Y^{ord}$ be the subset of all $\mathcal{O}_B$-linear points. Here is the Step 1 outlined above.

Proposition 4.9. *Let the notation and assumptions be as above. The subset Y^{lin} contains the closed points of an open dense subscheme U of Y^{ord}.*

PROOF.   We first describe the main steps of the proof.

$\mathcal{O}_B$-*linearity at y* : In view of Proposition 4.6, it follows that Y is $\mathcal{O}_B$-linear at y. In fact, we can suppose that $b = 1$ and $a = a_x$ is a unit in $\mathrm{End}_{\epsilon \mathcal{O}_B}(\widehat{S})$. We regard a_x as an isomorphism $a_x : \epsilon \mathcal{A}_{1,x}[p^\infty] \to \epsilon \mathcal{A}_{2,x}[p^\infty]$ of Barsotti-Tate groups. *Existence of U via extension of a homomorphism* : Based on Serre-Tate deformation theory, we reduce the existence of U to the existence of an étale covering $\widetilde{U}$ of an open dense subscheme of Y^{ord} containing y such that a_x extends to a homomorphism $\tilde{a} : \epsilon \mathcal{A}_1[p^\infty]_{/\widetilde{U}} \to \epsilon \mathcal{A}_2[p^\infty]_{/\widetilde{U}}$ of Barsotti-Tate groups over $\widetilde{U}$. *Extension of a homomorphism* : Based on Serre-Tate deformation theory, we first show that a_x extends to a homomorphism $\widehat{a} : \epsilon \mathcal{A}_1[p^\infty]_{/\widehat{Y}_y} \to \epsilon \mathcal{A}_2[p^\infty]_{/\widehat{Y}_y}$ of Barsotti-Tate groups over $\widehat{Y}_y$. The existence of $\widetilde{U}$ with the desired property then follows by an $fpqc$-descent.

We now describe each step.

$\mathcal{O}_B$-*linearity at y* : By part (1) of Proposition 4.6, the formal torus $\widehat{\mathcal{Y}}_y$ is canonically isomorphic to a

formal subtorus $\widehat{\mathbb{G}}_m \otimes L$ of $\widehat{S}^2 = \widehat{\mathbb{G}_m^2}$ for a $\mathbf{Z}_p$-free direct summand L of $\mathbf{Z}_p^2$. As $\dim \mathcal{X} = \dim V$, we conclude that the equation of $\widehat{\mathcal{Y}}_y$ in $\widehat{S}^2$ is given by $t_1^a = t_2^b$, for some $a, b \in \mathbf{Z}_p$ such that $a\mathbf{Z}_p + b\mathbf{Z}_p = \mathbf{Z}_p$. Moreover,

$$(4.5) \qquad\qquad L = \left\{ (c, d) \in \mathbf{Z}_p^2 : ac = bd \right\}.$$

In particular, Y is $\mathcal{O}_B$-linear at y. In view of (E), we replace (a, b) by $(a/b, 1)$ and suppose that $(a, b) = (a_x, 1)$ with $a_x \in \mathbf{Z}_p^\times$. We now normalise the action of $a \in \mathbf{Z}_p$ on $\widehat{S}$ as follows. We modify a by an element in $\mathbf{Z}_p$ if necessary, such that a acts as identity on $\epsilon A_x[p^\infty]^\circ$ without affecting the original action of a on $\widehat{S}$. In other words, a acts on $\widehat{S}$ via the action on $\epsilon A_x[p^\infty]^{\acute{e}t}$. We identify $\mathcal{A}_{i,y}$ with A_x. In this way, we regard a_x as an isomorphism $a_x : \epsilon \mathcal{A}_{1,x}[p^\infty] \to \epsilon \mathcal{A}_{2,x}[p^\infty]$ of Barsotti-Tate groups.

Existence of U via extension of a homomorphism : Suppose that there exists an open dense subscheme $U \subset Y^{ord}$ containing y having an irreducible étale cover $\tilde{U}$ such that a_x extends to an isomorphism $\tilde{a} : \epsilon \mathcal{A}_1[p^\infty]_{/\widetilde{U}} \to \epsilon \mathcal{A}_2[p^\infty]_{/\widetilde{U}}$ of Barsotti-Tate groups over $\widetilde{U}$. We show that Y is $\mathcal{O}_B$-linear at the closed points of U. In view of (E) and Lemma 4.7, shrinking the neighbourhood U of y if necessary, we suppose that the projection $\Pi_i : U \to V$ is étale. Let $u \in U$ and $(\Pi_1(u), \Pi_2(u)) = (u_1, u_2)$. In view of Serre-Tate deformation theory (cf. §3.1) and the isomorphism $\pi_i : \widehat{Y}_u \simeq \widehat{V}_{u_i}$, it follows that $\widehat{Y}_u$ is isomorphic to the deformation space of Barsotti-Tate $\epsilon \mathcal{O}_{B,p}$-module $\epsilon A_{u_i}[p^\infty]$ via Π_i. Note that the universal deformation of the Barsotti-Tate $\epsilon \mathcal{O}_{B,p}$-module $\epsilon A_{u_i}[p^\infty]$ is the Barsotti-Tate $\epsilon \mathcal{O}_{B,p}$-module $\epsilon \mathcal{A}_i[p^\infty]_{/\widehat{Y}_u} \simeq \epsilon \mathcal{A}[p^\infty]_{/\widehat{V}_{u_i}}$. We choose a p^∞-level structure $\eta_{i,p}$ on $\epsilon A_{u_i}[p^\infty]$ for $i = 1, 2$ (cf. (PL)). Accordingly, we have the Serre-Tate co-ordinates t_i (cf. (CC) and (STC)). The composition $\tilde{a} \circ \eta_{1,p}$ gives rise to a possibly new p^∞-level structure on $\epsilon A_{u_2}[p^\infty]$. In view of the fact that $\text{Aut}_{\epsilon \mathcal{O}_{B,p}} \widehat{Y}_u = \mathbf{Z}_p^\times$ and Definition 3.2, it follows that the new p^∞-level structures give rise to the Serre-Tate co-ordinate $t_2^{a_u}$ for some $a_u \in \mathbf{Z}_p^\times$. Thus, we have $t_1 = t_2^{a_u}$. In other words, $\widehat{Y}_u$ is contained in the formal subscheme of $\widehat{V}_{u_1} \times \widehat{V}_{u_2}$ defined by $t_1 = t_2^{a_u}$. As $\widehat{Y}_u$ is a smooth formal subscheme with an isomorphism $\pi_i : \widehat{Y}_u \simeq \widehat{V}_{u_i}$, we conclude that $\widehat{Y}_u$ is itself defined by the equation $t_1 = t_2^{a_u}$.

Extension of a homomorphism : We first extend a_x to a homomorphism $\widehat{a} : \epsilon \mathcal{A}_1[p^\infty]_{/\widehat{Y}_y} \to \epsilon \mathcal{A}_2[p^\infty]_{/\widehat{Y}_y}$ of Barsotti-Tate groups over $\widehat{Y}_y$. Let n be a positive integer. Let $\alpha_n \in H_x(\mathbf{Z}_{(p)})$ be a prime to p element such that $\alpha_n^{1-c} \equiv a \pmod{p^n}$. The homomorphism α_n also induces a homomorphism $\alpha_n : \mathcal{A}_{/V} \to \mathcal{A}_{/V}$ and hence, a homomorphism

$$\alpha_n : \epsilon \mathcal{A}_1[p^n]_{/\widehat{Y}} \xrightarrow[\Pi_1]{\sim} \epsilon \mathcal{A}[p^n]_{/\widehat{V}_x} \xrightarrow{\alpha_n} \epsilon \mathcal{A}[p^n]_{/\widehat{V}_x} \xrightarrow[\Pi_2^{-1}]{\sim} \epsilon \mathcal{A}_2[p^n]_{/\widehat{Y}}.$$

Thus, we have a homomorphism $\alpha_n : \epsilon \mathcal{A}_1[p^n]_{/\widehat{Y}_n} \to \epsilon \mathcal{A}_2[p^n]_{/\widehat{Y}_n}$ for an infinitesimal neighbourhood $\widehat{Y}_n$ of y. In view of Lemma 3.5, it follows that the homomorphism α_n equals $a|_{\epsilon \mathcal{A}_1[p^n]}$ over $\widehat{Y}_n$. Let $\widehat{a} = \varprojlim \alpha_n|_{\epsilon \mathcal{A}_1[p^n]_{/\widehat{Y}_n}}$. By definition, $\widehat{a} : \epsilon \mathcal{A}_1[p^\infty]_{/\widehat{Y}_y} \to \epsilon \mathcal{A}_2[p^\infty]_{/\widehat{Y}_y}$ is a homomorphism of Barsotti-Tate groups over $\widehat{Y}_y$. The existence of $\tilde{U}$ with the desired can be proven from the existence of $\tilde{a}$ by an $fpqc$-descent by an argument similar to [20, Prop. 8.4] and [26, pp. 92-93]. $\qquad\qquad\qquad \square$

Here is the Step 2 outlined in the introduction of this subsection.

Theorem 4.10. *Let X be as before. Then, X is smooth and there exist $\alpha, \beta \in H_x(\mathbf{Z}_{(p)})$ such that $X = \Delta_{\alpha,\beta}$. Moreover, if (E) holds, then we can take $(\alpha, \beta) = (1, \beta)$ with β being a p-unit.*

PROOF.   Recall that we have dominant projections $\Pi_i : Y \to V_i$. It follows that $\mathrm{End}(\mathcal{A}_i) \otimes \mathbf{Q} = B$. In view of the discussion before (E), we suppose that (E) holds.

Let $\mathcal{A}_{/Y} = \mathcal{A}_1 \times_Y \mathcal{A}_2$ and $\mathrm{End}^{\mathbf{Q}}(\mathcal{A}) = \mathrm{End}(\mathcal{A}_{/Y}) \otimes \mathbf{Q}$. We now have two possibilities for $\mathrm{End}^{\mathbf{Q}}(\mathcal{A})$, namely B^2 or $M_2(B)$. Based on de Jong's theorem in [22], we first show that $\mathrm{End}^{\mathbf{Q}}(\mathcal{A}) = M_2(B)$. Along with Serre-Tate deformation theory, we later finish the proof.

We first suppose that $\mathrm{End}^{\mathbf{Q}}(\mathcal{A}) = B^2$. We proceed in a way similar to the proof of Theorem 4.6. Let U be as in Proposition 4.9. In view of (E) and (PL), it follows that

$$(4.6) \qquad\qquad \mathrm{End}_{\epsilon \mathcal{O}_{B,p}} \epsilon \mathcal{A}[p^\infty]_{/U} = M_2(\mathcal{O}_{M_0}).$$

By Theorem 2.6 of [22], we have

$$(4.7) \qquad\qquad \mathrm{End}_{\mathcal{O}_{B,p}}(\mathcal{A}[p^\infty]_{/U}) = \mathrm{End}_{\mathcal{O}_B}(\mathcal{A}_{/U}) \otimes \mathbf{Z}_p.$$

This contradicts our assumption that $\mathrm{End}^{\mathbf{Q}}(\mathcal{A}) = B^2$. Thus, $\mathrm{End}^{\mathbf{Q}}(\mathcal{A}) = M_2(B)$.

Let $e_i \in \mathrm{End}^{\mathbf{Q}}(\mathcal{A})$ be two commuting idempotents such that $e_i(\mathcal{A}) = \mathcal{A}_{i/Y}$. As $\mathrm{End}^{\mathbf{Q}}(\mathcal{A}) = M_2(B)$, there exists $\tilde{\beta} \in GL_2(\mathcal{O}_{B,(p)})$ such that $\tilde{\beta} \circ e_1 = e_2$. Thus, we have as isogeny $\tilde{\beta} : \mathcal{A}_1 \to \mathcal{A}_{2/Y}$. In particular, there exists $g \in G(\mathbf{A}_{\mathbf{Q},f}^{(p)})$ such that $\varphi \circ \eta_1^{(p)} = \eta_2^{(p)} \circ g$. Specialising to the fiber of $\mathcal{A}_i$ at y, we conclude that g is induced by an element $\beta \in H_x(\mathbf{Z}_{(p)})$. Thus, the equation defining $\widehat{Y}_y \subset \widehat{S} \times \widehat{S}$ is given by $t_2 = t_1^{\beta^{1-c_x}}$. In view of (E) and the argument in $\mathcal{O}_B$-linearity step of the proof of Proposition 4.9, we conclude that $\beta^{1-c_x} \in \mathbf{Z}_{(p)}^\times$. Here c_x denotes the non-trivial element in the Galois group of the imaginary quadratic extension over the rationals associated to x. Thus, we suppose that $\beta \in \mathbf{Z}_p^\times$. Recall that in the proof of (1) of Proposition 4.6 we have $\widehat{X} = \bigcup_{L \in I} \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} L$, where L is an $\mathbf{Z}_p$-direct summand of $X_*(\widehat{S^2})$ and the union is finite. Moreover, the points of Y above x^2 are indexed by $L \in I$. More precisely, if y corresponds to L, then $\widehat{Y}_y = \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} L$. Thus, $\widehat{\Delta_{1,\beta,x^2}} = \widehat{Y}_y \subset \widehat{X}$. By an $fpqc$-descent, it follows that $\Delta_{1,\beta} \subset X$. As X is irreducible, we conclude that $\Delta_{1,\beta} = X$.   □

Thus, we have proven Theorem 4.5 for $n = 2$.

4.5. **Global structure II.** In this subsection, we study a class of global structure of locally stable subvarieties of n-fold self-product of the Shimura curve for $n \geq 2$.

Let the notation and hypotheses be as in §4.2.

Moreover, we suppose that $\mathcal{X} \neq S^n$.

Theorem 4.11. *Let X be as before. Then, X is smooth and there exist $\alpha, \beta \in H_x(\mathbf{Z}_{(p)})$ such that $X = V^{n-2} \times \Delta_{\alpha,\beta}$ up to a permutation of the first $n - 1$ factors.*

PROOF.   The proof of Proposition 4.9 also works for $n \geq 2$.

Let $\mathcal{A}_{/Y} = \mathcal{A}^n \times_{V^n} Y$ and $\mathrm{End}^{\mathbf{Q}}(\mathcal{A}) = \mathrm{End}(\mathcal{A}_{/Y}) \otimes \mathbf{Q}$. We now have two possibilities for $\mathrm{End}^{\mathbf{Q}}(\mathcal{A})$, namely B^n or $B^{n-2} \times M_2(B)$. In a same way as in the proof of Theorem 4.10, it can be shown that $\mathrm{End}^{\mathbf{Q}}(\mathcal{A}) = B^{n-2} \times M_2(B)$. Thus, there exists $i < n$ such that i^{th} factor $\mathcal{A}_i$ of $\mathcal{A}_{/Y}$ is isogenous to the last factor $\mathcal{A}_n$ of $\mathcal{A}_{/Y}$.

Based on Serre-Tate deformation theory, we finish the proof in the same way as the proof of Theorem 4.10.
□

This finishes the proof of Theorem 4.5.

Theorem 4.11 can also be proven by induction on n (cf. similar to the proof of [26, Cor. 3.19]).

4.6. **Linear independence.** In this subsection, we prove the linear independence of mod p modular forms as formulated in §4.1 based on the global structure of locally stable subvarieties.

As before, we have the following independence.

Theorem 4.12. *Let x be a closed ordinary point on the Shimura variety with the local stabiliser H_x. For $1 \leq i \leq n$, let $a_i \in H_x(\mathbf{Z}_p)$ such that $a_i a_j^{-1} \notin H_x(\mathbf{Z}_{(p)})$ for all $i \neq j$ (cf. §3.2). Let $f_1, ..., f_n$ be non-constant mod p modular forms on V. Then, $(a_i \circ f_i)_i$ are algebraically independent in the Serre-Tate deformation space $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$.*

PROOF.   From the definition of $\mathfrak{b}$ (cf. §4.1), it suffices to show that $\mathfrak{b} = 0$. This is equivalent to show that $X = V^n$.

When $n = 1$, this follows from the definition. Let $n \geq 2$. We first note that there exists y over x^n such that
$$\widehat{\mathcal{Y}}_y \supset \widehat{\Delta} := \left\{ (t^{a_1}, t^{a_2}, ..., t^{a_n}) | t \in \widehat{S} \right\}$$
(cf. similar to [26, pp.85]). When $n = 2$, this verifies the hypothesis in Theorem 4.5. When $n \geq 2$, the hypothesis in the theorem can be verified by induction on n up to a permutation of the first $n-1$ factors.

In view of the theorem, the locally stable subscheme X equals V^n or $V^{n-2} \times \Delta_{\alpha,\beta}$ for some $\alpha, \beta \in H_x(\mathbf{Z}_{(p)})$ up to a permutation of the first $n-1$ factors. We first suppose that $X = V^{n-2} \times \Delta_{\alpha,\beta}$. Let $\mathcal{X}'$ be the projection of $\mathcal{X}$ to the last two factors of S^n. As $X = V^{n-2} \times \Delta_{\alpha,\beta}$, the equation of $\widehat{\mathcal{X}'}_{x^2} \subset \widehat{S^2}$ is given by $t^{\beta^{1-c_x}} = (t')^{\alpha^{1-c_x}}$ for t (resp. t') being the Serre-Tate co-ordinate of the second last (resp. last) factor of $\widehat{S^n}$. On the other hand, it follows from the definition of X that the equation is also given by $t^{a_n} = t'^{a_{n-1}}$. Thus, $a_n a_{n-1}^{-1} = (\beta\alpha^{-1})^{1-c_x} \in H_x(\mathbf{Z}_{(p)})$. This contradicts our hypothesis on a_i's and finishes the proof.
□

Remark. The independence also plays a key role in [13]. In [13], we consider an (l, p)-analogue of the non-triviality of generalised Heegner cycles modulo p.

5. μ-invariant of Anticyclotomic Rankin-Selberg p-adic L-functions

In this section, we prove the vanishing of the μ-invariant of a class of anticyclotomic Rankin-Selberg p-adic L-functions. In §5.1, we consider the p-depletion of a normalised Jacquet-Langlands transfer of an elliptic Hecke eigenform. In §5.2, we describe generalities regarding the anticyclotomic Rankin-Selberg p-adic L-functions. In §5.3, we prove the vanishing of the μ-invariant.

5.1. p-**depletion.** In this subsection, we consider the p-depletion of a normalised Jacquet-Langlands transfer of an elliptic Hecke eigenform.

Let the notation and hypotheses be as in the introduction. Let $f \in S_k(\Gamma_0(N), \epsilon)$ be an elliptic newform such that:

(irr) the residual Galois representation ρ_f modulo p is absolutely irreducible.

The Jacquet-Langlands correspondence implies the existence of a classical modular form f_B on Sh_B such that the following holds:

(JL1) f_B is a classical modular form on the Shimura curve Sh_B of weight k and neben-character ϵ.
(JL2) For positive integer n such that $(n, N^-) = 1$, f_B is a Hecke eigenform for the Hecke operator T_n with the same eigenvalue as f.

We normalise f_B by requiring that

(5.1) $$\mu(f_B) = 0.$$

In what follows, we consider f_B as being defined over $\mathcal{O}_{E_f, \mathfrak{P}}$ and regard it as a p-adic modular form on the Shimura variety Sh. Here E_f is the Hecke field and $\mathfrak{P}$ a prime above p induced by ι_p.

We have the following useful lemma.

Lemma 5.1. *Let the notation and assumptions be as above. Then, the Hecke eigenform f_B is non-constant modulo p.*

PROOF.   If the Hecke eigenform f_B is constant modulo p, then the Hecke eigensystem is Eisenstein. This contradicts (irr).

$\square$

A class of anticyclotomic Rankin-Selberg p-adic L-functions is constructed via toric periods of the p-depletion of f_B.

Let

$$(5.2) \qquad f_B^{(p)} = f_B|_{VU-UV}$$

be the p-depletion of f_B. Here V and U are Hecke operators in [6, §3.6].

We recall the following lemma.

Lemma 5.2. *Let the notation be as above. Let* $f_B(t) = \sum_{\xi \in \mathbf{Z}_{\geq 0}} a(\xi, f_B) t^\xi$ *be the* t-*expansion of* f_B *around* x. *We have*

$$f_B^{(p)}(t) = \sum_{\xi \in \mathbf{Z}_{\geq 0}, p \nmid \xi} a(\xi, f_B) t^\xi.$$

PROOF. This readily follows from [6, Prop. 4.17]. $\qquad\qquad\square$

We have the following proposition regarding the p-integrality of the p-depletion.

Proposition 5.3. *Let the notation and assumptions be as above. We have*

$$\mu(f_B^{(p)}) = 0.$$

Moreover, the p-*depletion is non-constant modulo* p.

PROOF. The proof is based on the p-integrality criterion in [42, Prop. 2.9].

In view of the criterion and (5.1), we have

$$(5.3) \qquad \min_{M,\chi} \mu\left(\frac{L_\chi(f_B)}{\Omega_M^{2k}}\right) = 0.$$

Here M is an imaginary quadratic extension of $\mathbf{Q}$ such that it has an embedding into the indefinite quaternion algebra B, p splits in M and p does not divide the class number of M. Moreover, χ is an unramified Hecke character over M with infinity type $(k, 0)$, $L_\chi(f_B)$ the toric period of the pair (f, χ) and Ω_M the CM period (*cf.* [42, §2.3]).

As f_B is a Hecke eigenform, we have

$$(5.4) \qquad L_\chi(f_B^{(p)}) = (1 - \chi^{-1}(\bar{\mathfrak{p}})a_p + \chi^{-2}(\bar{\mathfrak{p}})\epsilon(p)p^{k-1})L_\chi(f_B)$$

(*cf.* [6, Prop. 8.9]). Here $\mathfrak{p}$ is a prime above p induced by the p-adic embedding ι_p as before and a_p denotes the T_p-eigenvalue of f.

It thus follows that

$$(5.5) \qquad \min_{M,\chi} \mu\left(\frac{L_\chi(f_B^{(p)})}{\Omega_M^{2k}}\right) = 0.$$

In view of the criterion, this finishes the proof.

"Moreover" part now immediately follows from the previous lemma.

$\qquad\qquad\square$

For later purposes, we introduce the following modular forms related to the p-depletion.

Let $\mathcal{U}_p$ be the torsion subgroup of $\mathbf{Z}_p^\times$. For $u \in \mathcal{U}_p$, let $\phi_u : \mathbf{Z}/p\mathbf{Z} \to \mathcal{W}$ be the indicator function corresponding to u.

Let $f_{B,u}$ be the modular form given by

(5.6) $$f_{B,u} = f_B | \phi_u.$$

Lemma 5.4. *Let the notation and assumptions be as above. We have*

(5.7) $$f_B^{(p)} = \sum_{u \in \mathcal{U}_p} f_{B,u}.$$

PROOF. This follows from the t-expansion principle, Proposition 3.6 and Lemma 5.2.

$\square$

5.2. **Anticyclotomic Rankin-Selberg p-adic L-functions.** In this section, we describe generalities regarding a class of anticyclotomic Rankin-Selberg p-adic L-functions. This is a slight reformulation of the results in [6, §8]. We adapt the formulation in [28, §1.3.7], [5, §8] and [32].

Let the notation and hypotheses be as in the introduction. Let $\Gamma_{N^+} = \varprojlim G_{p^n}$, where $G_{p^n} = \mathrm{Gal}(H_{N^+p^n}/K)$. Recall that Γ denotes the $\mathbf{Z}_p$-quotient of Γ_{N^+}. We fix a section of the projection $\pi : \Gamma_{N^+} \twoheadrightarrow \Gamma$ stable under the action of complex conjugation c. Let $\mathcal{C}(\Gamma, \overline{\mathbf{Z}}_p)$ be the space of continuous functions on Γ with values in $\overline{\mathbf{Z}}_p$.

Let Σ_{cc} denote the set of Hecke character over K central critical for f i.e. the set of Hecke characters λ such that λ of infinity type (j_1, j_2) with $j_1 + j_2 = k$ and $\epsilon_\lambda = \epsilon_f \mathbf{N}^k$. Let $\Sigma_{cc}^{(1)}$ be the subset of Hecke character of K with infinity type (l_1, l_2) such that $1 \le l_1, l_2 \le k - 1$. Let $\Sigma_{cc}^{(2)}$ be the subset of Hecke character over K with infinity type (l_1, l_2) such that $l_1 \ge k$ and $l_2 \le 0$. Recall that $\mathfrak{X}$ denotes the set of anticyclotomic Hecke characters over K factoring though Γ.

For $\chi \in \Sigma_{cc}^{(1)}$ (resp. $\Sigma_{cc}^{(2)}$), the global root number of the Rankin-Selberg convolution $L(f, \chi^{-1}, s)$ equals -1 (resp. 1). From now, we fix an unramified Hecke character $\chi \in \Sigma_{cc}^{(2)}$ with infinity type $(k, 0)$. For $\eta \in \Gamma$, note that $\chi\eta \in \Sigma_{cc}^{(2)}$.

Let

$$Cl_- := K^\times \mathbf{A}_{\mathbf{Q},f}^\times \backslash \mathbf{A}_{K,f} / U_K$$

and Cl_-^{alg} the subgroup of Cl_- generated by ramified primes. Here $U_K = (K \otimes_\mathbf{Q} \mathbf{R})^\times \times (\mathcal{O} \otimes_\mathbf{Z} \widehat{\mathbf{Z}})^\times$. Let $\mathcal{R}$ be the subgroup of $\mathbf{A}_K^\times$ generated by $K_v^\times$ for ramified v. Let $Cl_-^{alg} \subset Cl_-$ be the subgroup generated by $\mathcal{R}$. Let $\mathcal{D}_1$ be a set of representatives for Cl_-/Cl_-^{alg} in $(\mathbf{A}_{K,f}^{(pN)})^\times$. Let $\mathcal{U}^{alg} = U_K \cap (K^\times)^{1-c}$.

For $a \in \mathcal{D}_1$, let $\mathcal{F}_B^{(p)}(x(a))$ be the p-adic measure on Γ such that

$$\int_\Gamma \binom{x}{n} d\mathcal{F}_B^{(p)}(x(a)) = \binom{d}{n} f_B^{(p)}(x(a)).$$

Here n is a non-negative integer and d denotes the Katz p-adic differential operator.

We have the following useful result.

Lemma 5.5. *The power series expansion of the measure $\mathcal{F}_B^{(p)}(x(a))$ regarded as a p-adic measure on $\mathbf{Z}_p$ with support in $1+p\mathbf{Z}_p$ equals the t-expansion of the p-depletion $f_B^{(p)}$ around the CM point $x(a)$.*

PROOF. This follows from a similar argument as in the proof of [5, Prop. 8.1]. □

Let $L_p(f,\chi)$ be the p-adic measure on $\Gamma_{\mathfrak{N}^+}$ such that for $\varphi \in \mathcal{C}(\Gamma, \overline{\mathbf{Z}}_p)$, we have

$$(5.8) \qquad \int_{\Gamma_{N^+}} \varphi dL_p(f,\chi) = \sum_{a\in\mathcal{D}_1} \chi(a) \int_\Gamma \varphi|[a]df_B^{(p)}(x(a)).$$

The operator $|[a] \in \operatorname{End}(\mathcal{C}(\Gamma, \overline{\mathbf{Z}}_p))$ is given by $\varphi \mapsto \varphi|[a](\sigma) = \varphi(\sigma\operatorname{rec}_K(a)|_\Gamma)$.

To state the interpolation property of the measure, we need further notation.

Let

$$(5.9) \qquad \alpha(f, f_B) = \frac{\langle f, f\rangle}{\langle f_B, f_B\rangle}.$$

Here $\langle\cdot,\cdot\rangle$ denotes the normalised Petersson inner product in [42, §1].

For an unramified Hecke character $\lambda \in \Sigma_{cc}^{(2)}$ with infinity type $(k+j, -j)$ for $j \geq 0$, let

$$(5.10) \qquad C(f,\lambda) = \frac{1}{4}\pi^{k+2j-1} \cdot \Gamma(j+1)\Gamma(k+j) \cdot w_K\sqrt{|d_K|}2^{|S_f|} \cdot \prod_{l|N^-} \frac{l-1}{l+1}.$$

Here S_f denotes the set of primes which ramify in K that divide N^+ but do not divide the conductor of ϵ.

Let $\mathfrak{b} \subset \mathcal{O}_K$ be an ideal, $b_N \in \mathcal{O}_K$ and $W_f \in \mathbf{C}^\times$ as in [6, Prop. 8.3] and following Corollary 8.4 of [6], respectively.

Let

$$(5.11) \qquad W(f,\lambda) = W_f \cdot \epsilon_f(\mathbf{N})(\mathfrak{b})^{-1}\lambda\mathbf{N}^{-j}(\mathfrak{b}) \cdot (-N)^{k/2+j}b_N^{-k-2j}.$$

Let $(\Omega, \Omega_p) \in \mathbf{C}^\times \times \mathbf{C}_p^\times$ be the complex and p-adic CM periods in the beginning of [6, §8.4].

We have the following result regarding the p-adic variation of central critical Rankin-Selberg L-values over the $\mathbf{Z}_p$-anticyclotomic extension of K.

Theorem 5.6. *(Brooks) Let the notation be as above. Let $f \in S_k(\Gamma_0(N), \epsilon)$ be an elliptic newform and $\chi \in \Sigma_{cc}^{(2)}$ an unramified Hecke character over K with infinity type $(k, 0)$. For an unramified Hecke character $\nu \in \mathfrak{X}$ with infinity type $(m, -m)$, we have*

$$\frac{\widehat{\nu}(L_p(f,\chi))}{\Omega_p^{2(k+2(a+m))}} = (1 - (\chi\nu)^{-1}(\overline{\mathfrak{p}})a_p + (\chi\nu)^{-2}(\overline{\mathfrak{p}})\epsilon(p)p^{k-1}) \cdot t_K \cdot \frac{C(f,\chi\nu)}{\alpha(f, f_B)W(f,\chi\nu)} \cdot \frac{L(f,\chi\nu, 0)}{\Omega^{2(k+2m)}}.$$

Here

$$t_K = \frac{|\mathcal{U}^{alg}|}{[\mathcal{O}^\times : \mathbf{Z}^\times]|Cl_-^{alg}|}$$

PROOF.   This is essentially proven in [6, Prop. 8.9] based on the Waldspurger formula on the Shimura curve. The extra factor t_K arises from the definition (5.8).

□

Remark. (1). Note that t_K equals a power of 2.

(2). We have an analogous construction of the p-adic L-function for unramified Hecke characters χ over K with infinity type $(k + a, -a)$ for $a \geq 0$.

5.3. μ-**invariant.** In this subsection, we prove the vanishing of the μ-invariant of a class of antiyclcotomic Rankin-Selberg p-adic L-function.

Let the notation and hypotheses be as in §5.2. Let Γ' be the open subgroup of Γ generated by the image of $1 + p\mathbf{Z}_p$ via rec_K. Let $\pi_- : (\mathbf{A}_{K,f}^{(pN)})^\times \to \Gamma$ be the map arising from the reciprocity law. Let $Z' := \pi_-^{-1}(\Gamma')$ be the subgroup of $(\mathbf{A}_{K,f}^{(pN)})^\times$ and let $Cl'_- \supset Cl_-^{alg}$ be the image of Z' in Cl_- and let $\mathcal{D}'_1$ (resp. $\mathcal{D}''_1$) be a set of representatives of Cl'_-/Cl_-^{alg} (resp. Cl_-/Cl'_-) in $(\mathbf{A}_{K,f}^{(pN)})^\times$. Let $\mathcal{D}_1 := \mathcal{D}''_1\mathcal{D}'_1$ be a set of representatives of Cl_-/Cl_-^{alg}. Let $\mathcal{D}_0$ be a set of representatives of $\mathcal{U}_p/\mathcal{U}^{alg}$ in $\mathcal{U}_p$.

We have the following theorem regarding the μ-invariant of the anticyclotomic Rankin-Selberg p-adic L-functions.

Theorem 5.7. *Let $f \in S_k(\Gamma_0(N), \epsilon)$ be an elliptic newform and $\chi \in \Sigma_{cc}^{(2)}$ an unramified Hecke character over K with infinity type $(k, 0)$. Suppose that the hypotheses (ord), (H1), (H2), (H3) and (irr) hold. Then, we have*

$$\mu(L_p(f, \chi)) = 0.$$

PROOF.   Let t denote the Serre-Tate co-ordinate of the deformation space of the CM point $x(1)$ corresponding to the trivial ideal class. For $a \in \mathcal{D}'_1$, let $\langle a \rangle_\Sigma$ be the unique element in $1 + p\mathbf{Z}_p$ such that $rec_{\Sigma_p}(\langle a \rangle_\Sigma) = \pi_-(rec_\mathcal{K}(a)) \in \Gamma'$.

For $(a, b) \in \mathcal{D}_1 \times \mathcal{D}''_1$, let

(5.12)
$$\widetilde{\mathcal{F}}(t) = \sum_{u \in \mathcal{U}_p} f_{B,u}(t^{u^{-1}})$$

and

(5.13)
$$\mathcal{F}^b(t) = \sum_{a \in b\mathcal{D}'_1} \chi(ab^{-1})\widetilde{\mathcal{F}}|[a](t^{\langle ab^{-1} \rangle}).$$

Here $|[a]$ is the Hecke action induced by a.

Let $L_p^b(f, \chi)$ be the restriction of the measure $L_p(f, \chi)$ to $\pi_-(b)\Gamma'$. By definition, we have

$$(5.14) \qquad \mu(L_p(f, \chi)) = \min_{b \in \mathcal{D}_1''} \mu(L_p^b(f, \chi)).$$

In view of the Lemma 5.5, we have

$$(5.15) \qquad d^q \mathcal{F}^b \big|_{t=1} = \int_{\Gamma'} \nu_q dL_p^b(f, \chi).$$

Here q is a non-negative integer and ν_q the p-adic character of Γ' such that $\nu_q(\mathrm{rec}_K(y)) = y^q$ for $y \in 1 + p\mathbf{Z}_p$.

It follows that the formal t-expansion $\mathcal{F}^b(t)$ equals the power series expansion of the measure $L_p^b(f, \chi)$ regarded as a p-adic measure on $\mathbf{Z}_p$ with support in $1 + p\mathbf{Z}_p$ (*cf.* (1.3)).

Note that

$$(5.16) \qquad \widetilde{\mathcal{F}}(t) = |\mathcal{U}^{alg}| \cdot \sum_{u \in \mathcal{U}_p/\mathcal{U}^{alg}} f_{B,u}(t^{u^{-1}}).$$

Thus, we have

$$(5.17) \qquad \mathcal{F}^b(t) = |\mathcal{U}^{alg}| \cdot \sum_{(u,a) \in \mathcal{D}_0 \times b\mathcal{D}_1'} \chi(ab^{-1}) f_{B,u}|[a](t^{\langle ab^{-1} \rangle u^{-1}}).$$

Note that $p \nmid |\mathcal{U}^{alg}|$. From [31, Lemma 5.3], the linear independence of mod p modular forms (cf. Theorem 4.1) and the t-expansion principle of p-adic modular forms, it follows that

$$(5.18) \qquad \mu(L_p^b(f, \chi)) = \min_{u \in \mathcal{D}_0} \mu(f_{B,u}).$$

In view of Proposition 5.3 and Lemma 5.4, this finishes the proof.

$\square$

Remark. (1). The hypothesis that the Hecke character χ is unramified is present in [6]. It mainly arises as Prasanna's explicit version of the Waldspurger formula in [42] is conditional on the hypothesis. It seems likely that the hypothesis can be removed from the above theorem once we have an explicit version of the Waldspurger formula in the ramified case. Under mild hypotheses, such a Waldspurger formula is perhaps available (*cf.* [50]).

(2). The theorem has a similar flavour as the results on the vanishing of the μ-invariant in [47] and [32]. In these articles, the hypothesis (irr) is essential for the vanishing. For a discussion of the necessity, we refer to the introduction of these articles.

(3). A closely related p-adic L-function is constructed in [36]. Our strategy applies to this p-adic L-function as well and we can deduce the vanishing of its μ-invariant under the above hypotheses.

(4). The theorem can be used as an input in the proof of Perrin-Riou's conjecture on Heegner points under mild hypotheses in [49]. The result on the μ-invariant in [32] is originally used in [49].

6. NON-TRIVIALITY OF p-ADIC ABEL-JACOBI IMAGE MODULO p

In this section, we consider the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p. In §6.1, we describe the p-adic Waldspurger formula due to Brooks. In §6.2, we prove the non-triviality.

6.1. **p-adic Waldspurger formula.** In this subsection, we describe the p-adic Waldspurger formula due to Brooks relating certain values of the anticyclotomic Rankin-Selberg p-adic L-function outside the range of interpolation to the p-adic Abel-Jacobi image of generalised Heegner cycles.

Unless otherwise stated, let the notation and hypotheses be as in the introduction. Let f be a normalised elliptic newform of even weight $k \geq 2$, level $\Gamma_0(N)$ and neben-character ϵ. We also denote the Jacquet-Langlands transfer as in §5.1 by the same notation. Let ω_f be the corresponding differential.

We first recall that the construction of generalised Heegner cycles in [6] requires the weight being even (*cf.* [6, §6]).

Let

$$(6.1) \qquad\qquad r = \frac{k-2}{2}.$$

Let A be the CM abelian surface corresponding to the trivial ideal class in $\mathrm{Pic}(\mathcal{O})$ defined over the Hilbert class field H of K (*cf.* §2.4). Let $\mathcal{A}_r$ be the Kuga-Sato variety given by r-fold fiber product of the universal abelian surface. For an extension F/K containing the real quadratic field M_0 (*cf.* §2.1), let W_r be the variety over F given by $W_r = \mathcal{A}_r \times A^r$. By the abuse of notation, let ϵ also denote the idempotent in the ring of correspondences on W_r defined in [6, §6.1].

For an integer j such that $0 \leq j \leq 2r$, let $\omega_f \wedge \omega_A^j \eta_A^{2r-j} \in Fil^{2r+1} \epsilon H_{dR}^{4r+1}(W_r/F)$ be as in [6, §6.4]. Let n be a positive integer. For an ideal $\mathfrak{a} \subset \mathcal{O}_{N^+ p^n}$, let $\Delta_\mathfrak{a} \in \epsilon CH^{2r+1}(W_r \otimes L)_{0,\mathbf{Q}}$ be the codimension-$(2r+1)$ homologous to zero generalised Heegner cycle defined in [6, §6.2]. Here L is the field of definition of the cycle and $CH^{2r+1}(W_r \otimes L)_{0,\mathbf{Q}}$ is the Chow group of codimension-$(2r+1)$ homologous to zero cycles over L with rational coefficients.

Let

$$(6.2) \qquad\qquad AJ_p : \epsilon CH^{2r+1}(W_r)_{0,\mathbf{Q}} \to (Fil^{2r+1} \epsilon H_{dR}^{4r+1}(W_r/F)(r))^\vee$$

be the p-adic Abel-Jacobi map in [6, §6.3].

We have the following p-adic Waldspurger formula.

Theorem 6.1. *(Brooks) Let the notation be as above. Let $f \in S_k(\Gamma_0(N), \epsilon)$ be an elliptic newform and $\chi \in \Sigma_{cc}^{(2)}$ an unramified Hecke character over K with infinity type $(k, 0)$. Let η be a Hecke character such that $\chi\eta \in \Sigma_{cc}^{(1)}$ is an unramified Hecke character over K with infinity type $(k-1-j, 1+j)$ for $0 \leq j \leq 2r$. Let $\nu \in \mathfrak{X}_0$ be a primitive Hecke character of conductor p^n, where $n \geq 1$. Suppose that the hypotheses (ord), (H1), (H2) and (H3) hold. Then, we have*

$$\frac{\widehat{\eta\nu}(L_p(f,\chi))}{\Omega_p^{2(2r-2j)}} = \left(\frac{G(\nu^{-1})}{j!} \cdot \sum_{[\mathfrak{a}]\in G_{p^n}} (\chi\eta\nu)^{-1}(\mathfrak{a})\mathbf{N_K}(\mathfrak{a}) \cdot AJ_p(\Delta_{\mathfrak{a}})(\omega_f \wedge \omega_A^j \eta_A^{r-j})\right)^2.$$

PROOF.   This follows from the argument in the proof of Theorem 8.11 in [6]. As η is of p-power conductor, we obtain the extra factor of the Gauss sum. For a related argument, we refer to the proof of [18, Thm. 4.9].

$\square$

6.2. **Non-triviality.** In this subsection, we consider the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p.

Let the notation and hypotheses be as in §6.1. We have the following result regarding the non-triviality.

Theorem 6.2. *Let $f \in S_k(\Gamma_0(N), \epsilon)$ be an elliptic newform of even weight and $\chi \in \Sigma_{cc}^{(2)}$ an unramified Hecke character over K with infinity type $(k, 0)$. Let η be a Hecke character such that $\chi\eta \in \Sigma_{cc}^{(1)}$ is an unramified Hecke character over K with infinity type $(k-1-j, 1+j)$ for $0 \leq j \leq 2r$. Suppose that the hypotheses (ord), (H1), (H2), (H3) and (irr) hold. Then, we have*

$$\liminf_{\nu\in\mathfrak{X}_0} v_p\left(\frac{G(\nu^{-1})}{j!} \cdot \sum_{[\mathfrak{a}]\in G_{p^n}} (\chi\eta\nu)^{-1}(\mathfrak{a})\mathbf{N_K}(\mathfrak{a}) \cdot AJ_p(\Delta_{\mathfrak{a}})(\omega_f \wedge \omega_A^j \eta_A^{r-j})\right) = 0,$$

where p^n is the conductor of ν. Moreover, the same conclusion holds when $\mathfrak{X}_0$ is replaced by any of its infinite subset.

PROOF.   This follows readily from Theorem 5.7 and Theorem 6.1.

$\square$

Remark. (1). It follows that the generalised Heegner cycles are non-trivial in the top graded piece of the coniveau filtration on the Chow group over the $\mathbf{Z}_p$-anticyclotomic extension. The non-triviality can be seen as an evidence for the refined Bloch-Beilinson conjecture as follows. Recall that the Rankin-Selberg convolution in consideration is self-dual with root number -1. In view of [44] and the Jacquet-Langlands correspondence, the corresponding Galois representation contributes to an étale cohomology of W_r. The conjecture thus predicts the existence of a non-trivial cycle in the top graded piece of the coniveau filtration on the Chow group. Generalised Heegner cycles are a natural source of cycles in the setup and in the case of weight two, they coincide with classical Heegner points. We can thus expect a generic non-triviality of generalised Heegner cycles. For the details and the role of coniveau filtration, we refer to [3, §1 and §2] and [12].

(2). In view of the theorem and the construction of generalised Heegner cycles, it follows that the Griffiths group $Gr^{r+1}(W_{r/\overline{\mathbf{Q}}}) \otimes \mathbf{Q}$ has infinite rank. An analogous result for the Griffiths group of the Kuga-Sato variety $\mathcal{A}_r$ is due to Besser (*cf.* [4]). The approach in [4] is via consideration of

generic non-triviality of classical Heegner cycles over a class of varying imaginary quadratic extensions.

We have the following immediate corollary.

Corollary 6.3. *Let $f \in S_2(\Gamma_0(N), \epsilon)$ be an elliptic newform and χ be an unramified finite order Hecke character over K such that $\chi \mathbf{N_K} \in \Sigma_{cc}^{(1)}$. Suppose that the hypotheses (ord), (H1), (H2), (H3) and (irr) hold. Then, we have*

$$\liminf_{\nu \in \mathfrak{X}_0} v_p\bigg(G(\nu^{-1}) \log_{\omega_{B_f}}(P_f(\chi \nu)) \bigg) = 0.$$

In particular, for $\nu \in \mathfrak{X}_0$ with sufficiently large p-power order the Heegner points $P_f(\chi \nu)$ are non-zero in $B_f(H_{\chi \nu}) \otimes_{T_f} E_{f, \eta \nu}$. Moreover, the same conclusions hold when $\mathfrak{X}_0$ is replaced by any of its infinite subset.

PROOF.   This follows from the weight 2 case of the theorem.

$\square$

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